Theory of ratios in Nicomachus' *Arithmetica* and series of arithmetical problems in Pachymeres' *Quadrivium*: Reflections about a possible relationship

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Abstract. The voluminous *Treatise of the four mathematical sciences* of Georgios Pachymeres is the most renowned quadrivium produced in Byzantium. Among its specific features, historians of mathematics have pointed out, is the inclusion of Diophantus, besides Nicomachus and Euclid, in the sources for the arithmetical section and, accordingly, the incorporation of series of problems and problem-solving in its contents. The present paper investigates the “Diophantine portion” of Pachymeres’ treatise and it shows that it is structured according to two criteria intrinsically characterized by *seriality*: on one hand, the arrangement in which the problems are presented in book I of Diophantus’ *Arithmetica*; on the other hand, for those problems of which the enunciation involves ratio, the order in which Nicomachus discusses the kinds of ratios in his *Arithmetical introduction*. Furthermore, it analyses the solutions that Pachymeres offers and argues that Nicomachus’ *Arithmetical introduction* provides the necessary tools for pursuing them.

Résumé. Le volumineux *Traité des quatre sciences mathématiques* de Georgios Pachymère est le quadrivium plus connu produit à Byzance. Les historiens des mathématiques ont déjà remarqué que Pachymère s’est basé pour la section concernant l’arithmétique non seulement sur Nicomaque et Euclide, mais aussi sur Diophante, en incorporant ainsi une série de problèmes avec ses solutions. Le présent article examine la « partie diophantienne » du traité de Pachymère et il montre que cette partie est structurée selon deux critères intrinsèquement caractérisés par la *sérialité* : d’un côté, l’arrangement selon lequel les problèmes sont présentés dans le premier livre des *Arithmétiques* de Diophante ; de l’autre côté, pour les problèmes dont l’énoncé implique un rapport, l’ordre dans lequel Nicomaque traite dans son *Introduction arithmétique* les types de rapports. En outre, l’article analyse les solutions proposées par Pachymère et il suggère que le traité de Nicomaque fournit les outils nécessaires pour les réaliser.

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1. The restoration of the Byzantine Empire, after the recapture in 1261 of Constantinople by the Byzantines, was a turning point in the intellectual history of Byzantium. Conventionally, this year marks the beginning of a remarkable literary flourishing, referred to in the bibliography as “Palaeologan Renaissance”. In Steven Runciman’s words “At no other epoch was Byzantine society so highly educated and so deeply interested in things of the intellect and the spirit” (Runciman 1970, 2). The peak of this renaissance occurred under the rule of Andronikos II (1282–1328), the second emperor of the Palaeologan dynasty. However, the revival of the scholarly activity in the decades around the year 1300 did not arise out of the blue. Its roots are clearly recognised in the first half of the thirteenth century, especially in the Empire of Nicaea, the Greek state which was founded in the western part of Asia Minor after the conquest of Constantinople by the Fourth Crusade, and flourished during the Latin occupation of the capital (1204–1261).

The disasters that followed the siege of Constantinople in 1204 deeply affected the intellectual life of the capital. The institutional basis of education and learning was destroyed, libraries were dispersed, books were lost, eminent scholars, well known until 1204, disappeared thereafter for ever (Fryde 2000, 3). During this period the state of Nicaea became the political, religious and cultural centre of the Byzantine world. The elite who had fled from Constantinople after the capture, and found refuge there, not only provided the means for establishing a centralised administration (Constantinides 1982, 5), but also ensured the continuity of the Byzantine tradition, which is what characterises the Empire of Nicaea. This stands not only in the political, military, and ecclesiastical sphere, but in higher education as well, which was conceived and reorganised after the Constantinopolitan model (Browning 1960, 12). Thus, we find a number of learned men in this period who, encouraged by imperial patronage, kept higher learning alive, by acting as teachers, collecting books and creating libraries, generally by pursuing scholarly activities (Constantinides 1982, 159). The career and intellectual life of Nikephoros Blemmydes (1197–1269), the leading scholar of the Nicaean Empire, illustrate this very well.

Some aspects of the scholarly profile and activities of Blemmydes are particularly interesting for our purposes, since they prefigure analogous attitudes of the scholars of the early Palaeologan period, when the activity of Georgios Pachymeres took place. For this reason it is useful to mention these aspects here.

First, one must stress the many-sided character of the learning of these scholars. Although most of them were connected with the Church, secular learning was always part of their curriculum. This learning included not only poetry, grammar, philosophy and rhetoric, but the mathematical sciences as well. The Byzantine scholars of the period were in fact polymaths. In particular, Blemmydes’ involvement in the study of mathematics is quite interesting. Having received the *enkyklios paideia* in Nicaea he wanted to continue his studies to more advanced subjects but he couldn’t find a teacher to teach him. To this end he crossed the border and went to the Latin area, where he was taught mathematics at the monastery of Skamandros by a teacher called Prodomos. Blemmydes says in his *Autobiography* that this teacher taught him the quadrivial sciences of arithmetic, basic plane and solid geometry, advanced geometry (apparently from Euclid’s *Data*), and astronomy, as well as optics, syllogistic and physics (Heisenberg 1896, 5.1–7; 55.10–11). As Constantinides remarks, this curriculum cannot be explained without presupposing that a rich library was available at Prodomos’ school (Constantinides 1982, 8; 137). Furthermore, for the purposes of this paper it is important to notice that referring in particular to arithmetic Blemmydes adds in his *Autobiography* that he learned it from the works of Nicomachus and Diophantus, the latter not in its totality, but what his teacher had mastered of it (οὐ γὰρ πάσης, ἀλλ’ ὅσης ἦν εἰδήμων ὁ ἐκδιδάσκων, ὡς ἔλεγεν) (Heisenberg 1896, 5.1–4). This phrase indicates of course that Blemmydes was to some extent aware of Diophantus’ mathematics. It also entails that a manuscript of the *Arithmetica* existed in Prodomos’ school in the ’20s of the thirteenth century. At any rate, Blemmydes’ mention of Diophantus is the first we encounter in Byzantine sources after Psellos’ time (middle of the eleventh century). By the end of the thirteenth century, Diophantus

will be recognized as an outstanding authority in the field of arithmetic by the scholars of the early Palaeologan period.

Often, the polymathy of the scholars of the first half of the thirteenth century is combined with their constant preoccupation with the task of composing treatises intended to serve educational purposes.

Again Blemmydes' example illustrates this very well. His major philosophical work is an *Epitome*, in two books, one on Logic, the other on Natural philosophy. The educational purpose of the *Epitome* is manifested by the many examples it contains, thus making the material suitable for students, and by the ability of its author to present comprehensively the issues at hand (Zografidis 2011, 893). Quite rightly Fryde remarks that "the work is revealing about what would be regarded by contemporaries as a useful school manual" (Fryde 2000, 75).

The quest for books which, due to the disaster of 1204, were scattered or lost, and their subsequent concentration in schools and public libraries, was another task which preoccupied the scholars of the period. Blemmydes, for example, collected many rare books that he found during the great investigatory tour he made westwards, to the Aegean islands and Greece, and took them back to Asia Minor (Constantinides 1982, 134).

2. The aforesaid features of Nikephoros Blemmydes' intellectual activity can also be identified in the works of leading scholars of the Palaeologan period. Thus, polymathy, secular learning with a genuine awareness not only of the basic mathematical topics but of more advanced fields as well, teaching activities including the production of textbooks, quest for, gathering of, and reproduction of old manuscripts, are some of the characteristics of scholarly activity in Constantinople toward the end of the thirteenth century. In so far as the knowledge of mathematics is concerned, we know the names of at least a dozen scholars well-versed in mathematics and astronomy during the reigns of the first two Palaeologues (Fryde 2000, 338), the best known of whom being Georgios Pachymeres (1242 – ca. 1310) and Maximos Planoudes (1255–1305). In particular, the tasks of unearthing, restoring, and transcribing old manuscripts, and amassing them in libraries, reached during this period an unprecedented blooming. All these activities, which should be seen as parts of a wider movement activated by the vast influx of manuscripts from other regions to Constantinople after the recovery of the city in 1261 (Browning 1960, 12), are perfectly exemplified by Planoudes' occupation with the *Arithmetica* of Diophantus. From Planoudes' correspondence we know that, in circles around him, in the years 1292–1293 in Constantinople, there existed at least two manuscripts bearing the text of Diophantus. In order to prepare a new improved text of the *Arithmetica* Planoudes gathered and collated these manuscripts, the outcome of his efforts being the actual *Ambros. Et 157 sup.*, his autograph, of which only few folios survive (Turyn 1972, 78–80). However, Planoudes' interest in Diophantus should not be considered as an isolated case. Georgios Pachymeres, the other major scholar of the period, was also well acquainted with the mathematics of Diophantus. In his *Quadrivium* he made, as we shall see below, a fresh and unconventional use of Diophantus' text, by joining it with the text of Nicomachus – a most interesting and significant novelty. More generally, it would be hardly exaggerated to say that at the end of the thirteenth century, and the beginning of the fourteenth, there was a real interest about the work of Diophantus in Byzantium. The mere fact that the rich direct tradition of the Greek text of the *Arithmetica* (no less than thirty witnesses) stems from four independent manuscripts (*Marc. gr.* 308, *Vat. gr.* 191, *Vat. gr.* 304, *Matrit.* 4678) three of which were produced during this period (Acerbi 2013, 712) proves the veracity of the claim.¹

¹ The manuscript of Madrid, formerly assigned to the 13th century, is now dated in the second half of the 11th century (Pérez-Martín 2006, 438–441).

3. Georgios Pachymeres, a prolific writer, most learned and highly respected scholar and church official, composed his *Quadrivium* around 1296 (Golitsis 2008, 63–64), that is, in the culmination of the literate florescence under the second Palaeologue. However, his undertaking should not be seen simply as the outcome of the intellectual atmosphere in Constantinople in the late thirteenth century. Nor should it be explained solely by the florescence, more specifically, of the mathematical learning in this period. Pachymeres' mathematical interests should also be understood in the background of his education, the lineage of which leads back to Nikephoros Blemmydes, through his pupil Georgios Akropolites (2008, 53). It is reasonable, therefore, to assume that Blemmydes' active interest in the quadrivial sciences, on the one hand, and in producing textbooks to serve educational purposes, on the other, was handed down to his direct and indirect disciples. From such a viewpoint Pachymeres' initiative to compose the *Quadrivium* is perfectly understandable. Furthermore, and this is more important for the history of mathematics, in this manner one can explain some of the specific features of the *Quadrivium*, namely the inclusion of Diophantus in the sources for the book on arithmetic and, accordingly, the incorporation in its contents of series of problems and problem-solving.

The full title of the work is “Treatise on the four mathematical sciences: arithmetic, music, geometry, and astronomy” (Σύνταγμα τῶν τεσσάρων μαθημάτων: ἀριθμητικῆς, μουσικῆς, γεωμετρίας καὶ ἁστρονομίας), however it is best known with the concise title « *Quadrivium* ». The text is preserved in at least fourteen manuscripts –the most important of which, *Angelicus* 38 (C. 3. 7), has been identified as an autograph of Pachymeres himself (Harlfinger 1971, 357 n. 3)–, and its complete edition was prepared by P. Tannery but published posthumously by E. Stéphanou (Tannery & Stéphanou 2007; references to this work will be given by the page and the line numbers).

According to Fryde (2000, 339), Pachymeres' *Quadrivium* is a witness to the adoption by Byzantines of the western, Latin scheme of the seven liberal arts that went back to Martianus Capella (5th century A.D.). In our view this claim must be slightly modified so as to become more accurate. In fact, the four mathematical disciplines were part of the higher education studies at least in the schools of the big cities of the late antique world, as preparatory material for the study of philosophy. This teaching is recognized in the history of Greek and Roman education, continuing almost uninterrupted into the education of the Eastern Empire, even during the times of great educational changes of the 9th and 10th century. The level of the learning in several periods and places varied, but this fact put aside, what really differentiates the western tradition from the eastern was first and foremost the lack, in Byzantium, of specialised textbooks. In contrast, the composition of such kind of books was widespread in the Latin West throughout the Middle Ages, and this claim is what must be acknowledged as the main contribution of Martianus Capella in the western tradition. Obviously though it was Boethius' crucial rendition of the compilation of the four disciplines that dominated mathematical learning in the Latin West, at least until the High Middle Ages. It seems that the first work of such kind produced in Byzantium was a concise anonymous *Quadrivium* composed in the beginning of the eleventh century (Heiberg 1921). Compared with it, Pachymeres' *Quadrivium* is significantly different, in every aspect more complete, scholarly and epistemically superior.

Pachymeres, as pointed out, was already acknowledged by his contemporaries for being highly learned in many fields (πολύιστορ). The *Quadrivium* clearly reflects Pachymeres' erudition, since its composition relies on a vast variety of sources to which the author had access. Thus, Constantinides (1982, 157), who praises the work for being “probably the best surviving Byzantine compilation on the four disciplines” writes quite aptly that “Pachymeres' *Quadrivium* can be regarded as an encyclopaedia based on ancient works”. The sources from which Pachymeres drew are the following: For the book on arithmetic the main sources are Nicomachus, Diophantus and Euclid; for music, Ptolemy and Porphyry; for geometry, Euclid (not only the *Elements* but the *Optics* as well); and for astronomy, Aratos, Archimedes, Aristotle, Cleomedes, Euclid, Ptolemy and Theon. We see, therefore, that Pachymeres' *Quadrivium* comprises a significant portion of the core of the ancient tradition in the mathematical

Table 1.

Theorems from Euclid's book II	3	4	5	6	7	8	9	10
Chapters from Pachymeres' book on arithmetic	47	48	49	50	51	52	53	54

sciences. It must be noticed, however, that, besides erudition, what characterises Pachymeres' redaction is that he does not imitate slavishly his models, as Herbert Hunger has remarked (Hunger 1978, ii, 246). In this sense, Pachymeres' work differs profoundly not only from the anonymous byzantine *Quadrivium* of the eleventh century, but from many western quadrivia as well.

The relative independence of Pachymeres from his authoritative sources is probably better illustrated in the way he uses Euclid's *Elements* in his books on arithmetic and geometry. Thus, while one would expect to draw for the book on arithmetic only from the arithmetical books VII, VIII, IX of the *Elements*, Pachymeres includes also in it theorems from book II, which however have been interpreted arithmetically. Indeed the series of eight theorems 3–10 of book II of the *Elements*, arithmetically interpreted, constitute respectively the sections 47–54 of the book on arithmetic of the *Quadrivium*, as shown on the table (Table 1).

As an example we give below how Pachymeres presents in chapter 49 the arithmetical equivalent of Euclid's II.5. First, let us see what the theorem II.5 is: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half" (Heath 1908, I, 382). This purely geometrical theorem has been transformed by Pachymeres into the following arithmetical theorem:

"If the same even number be cut into equal and unequal <parts>, the oblong number (προμήκης) constituted from the unequal <parts> of the whole, plus the square number from the <number> between the sections of the <whole> number, is equal to the square number on the half <of the whole number>.

For example, let the < number> be 12; let the 12 be cut into the equal <parts> 6 and 6, and into the unequal <parts> 8 and 4. In between the sections is found the number 2, which, when subtracted from 6, and when added to 6, produces the section into unequal <parts> . So we say: four times 8, 32, and twice 2, 4, which is the square from the <number> in between the sections; together <make> 36, which is square, equal to the square on the half, which is 36, <square> from 6" (80.14–22).

In the same spirit, arithmetical counterparts of the theorems II.3–10 are incorporated into the book on arithmetic of the *Quadrivium*. But more than that: Arithmetical interpretations of several geometrical propositions of the *Elements* are also found within the book on geometry as well. Namely, in chapters 3 and 4 arithmetical interpretations of the theorems II.12–13 are given, while in chapter 2 the same happens for theorem I.47 (Pythagorean Theorem).

The arithmetical interpretation of propositions in geometry is by no means an innovation of Pachymeres. Such a blending of geometry with arithmetised mathematics was quite common from the period of late antiquity onwards. Evidence abounds in the commentaries that accompany Euclid's text in the manuscripts bearing the *Elements* (Heiberg-Stamatis 1977), and, most importantly, in the problem-solving tradition. The method of "ἐπιλογισμὸς ἐκ τῶν γεωμετρικῶν ἐφοδῶν or δείξεων", which was amply used for solving problems of mensuration in geometry and astronomy, is a characteristic example. Nevertheless in Pachymeres' *Quadrivium* we do find marks of novelty. For example, the choice to append seventeen theorems from Euclid's *Optics* to the book on geometry (313.13–328.21), could be seen as an indication of an author's freshness and undogmatic attitude in dealing with his sources. The same can be said for the idea to include in the book on arithmetic a series of problems originating in the first book of Diophantus' *Arithmetica* and, furthermore, to combine ingeniously this material with material provided by Nicomachus' *Arithmetica*.

This clever and unconventional intermixture of two sources that are so different will concern us in the rest of this article. Before embarking into this study, however, we cannot avoid pointing out that Euclid's

Optics and Diophantus' *Arithmetica* were, both, included in the courses that a few decades earlier, in the beginning of the century, Prodromos had taught to Nikephoros Blemmydes. It is not implausible, then, that Pachymeres' choices in dealing with Diophantus and Euclid emanate from Blemmydes' training and interests; in such a case they could be explained on account of the line of apprenticeship connecting Pachymeres with Blemmydes, through Georgios Akropolites. If so, Blemmydes' sayings about his studies in the school of Prodromos should be taken more seriously than most modern historians usually believe; moreover, the role of Prodromos in the transmission of the text of the *Arithmetica* should be considered.

4. The book on arithmetic of Pachymeres' *Quadrivium* consists of 74 chapters. The first five chapters contain a preface of philosophical content, speaking about the task of the scholar and the pursuit of happiness deriving from knowledge, and a philosophical discussion of the number and the arithmetic. The following chapters 6–24 constitute the main “Nicomachean” part of the book. Next comes the “Diophantine” part, which covers the chapters 25–44. After two chapters dealing with the square and oblong numbers (*ἑτερομήκεις*), the book concludes with chapters 47–74 which form its “Euclidean” part.

The “Diophantine” part of the book, as said before, covers 20 chapters.² In the first chapter Pachymeres presents the technical terms of the “arithmetical theory” that Diophantus exposes in his introduction to the *Arithmetica*, and the operations with these terms. Regarding this chapter it is interesting to point out that, although Diophantus presents in his introduction all this material as making part of a coherent machinery –the several parts of which are interconnected in a suitable manner so as to produce a general way to be followed for solving problems in arithmetic³– in Pachymeres' text not all parts of the machinery are exposed. For example, Pachymeres does not mention the equation that arises from the problem, according to the general method taught by Diophantus, nor its treatment (simplification and solution). It is manifest that the aims of the two authors in composing their works were not the same.

In chapters 26–44 Pachymeres solves a number of arithmetical problems corresponding to the eleven first problems of the first book of the *Arithmetica*. He thus creates a series of problems, structured according to two criteria: on one hand, the arrangement in which the problems are presented by Diophantus; on the other hand, for those problems of which the enunciation involves ratio, the order in which Nicomachus discusses, in his *Arithmetical introduction*, the kinds of ratios. Because of the second criterion, the series of problems in Pachymeres is much lengthier than the corresponding series in Diophantus. Indeed, the problems in Diophantus are eleven, while the series of problems that Pachymeres creates is developed in nineteen chapters, each chapter dealing with a different type of problem. The qualification “different” refers here not only to the generic enunciation of the problem but, in the case the enunciation involves ratio, to the instantiated enunciation as well. In such cases the distinction is made on the basis of the type of the involved ratio.

Let us see how this is carried out in a concrete example. The second problem of Diophantus asks a given number to be divided into two numbers having a given ratio (Tannery 1893–95, i, 16.24–25). This generic enunciation is followed in Diophantus by a single instantiation, in which the given number is 60 and the given ratio is triple (Tannery 1893–95, i, 16.26–27). Pachymeres discusses problems modelled

² This part of Pachymeres' *Quadrivium* was first published by Tannery in the second volume of Diophantus works (Tannery 1893–95, ii, 78–122).

³ Diophantus presents the components of the machinery not as distinctive, independent to one another elements, but as parts which form a unified whole, in other words, as consecutive stages of a unique general method. This is made clear by phrases of his text like “after that” (Tannery 1893–95, i, 14.11), which reveal that the parts of the machinery are interconnected to each other (Christianidis 2007, 291–299).

Table 2.

Chapter	Kind of ratio	Number of problems
27	Multiple (πολλαπλάσιος)	3
28	Superparticular (ἐπιμόριος)	6
29	Superpartient (ἐπιμερής)	3
30–31	Multiple superparticular (πολλαπλασιεπιμόριος)	5 + 3
32	Multiple superpartient (πολλαπλασιεπιμερής)	5
		25

Table 3.

Name of ratio	
<i>Ratios greater than unity</i>	<i>Ratios less than unity</i>
Multiple (A, B), with $A = nB$	Submultiple (A, B), with $B = nA$
Superparticular (A, B), with $A = B + \frac{1}{n} \times B$	Subsuperparticular (A, B), with $B = A + \frac{1}{n} \times A$
Superpartient (A, B), with $A = B + (m \times \frac{1}{m+n}) \times B$	Subsuperpartient (A, B), with $B = A + (m \times \frac{1}{m+n}) \times A$
Multiple superparticular (A, B), with $A = kB + \frac{1}{n} \times B$	Submultiple superparticular (A, B), with $B = kA + \frac{1}{n} \times A$
Multiple superpartient (A, B), with $A = kB + (m \times \frac{1}{m+n}) \times B$	Submultiple superpartient (A, B), with $B = kA + (m \times \frac{1}{m+n}) \times A$

on the aforesaid problem of Diophantus in six chapters of his book (27–32). The division in chapters is made on the basis of the kind of the ratio, and in each chapter he solves more than one problem. The whole structure is shown in Table 2.

For the reader of this article who is not accustomed with the meaning of the terms appearing in the above table it is useful to remind at this point that the Nicomachean theory of ratios comprises five kinds of ratios, to which correspond five more kinds of inverse ratios (sub-contrary). The five couples are:

Multiple / Submultiple,
 Superparticular / Subsuperparticular,
 Superpartient / Subsuperpartient,
 Multiple superparticular / Submultiple superparticular, and
 Multiple superpartient / Submultiple superpartient.

The meaning of these highly cumbersome terms can be easily grasped by using modern symbolism, as shown in Table 3, in which the letters A, B, k , m , n denote positive integers, while the ratio of A to B is written (A, B):

Going back to Table 2 we see that from just one problem of Diophantus Pachymeres generates 25 problems. The pattern with which these problems have been created reveals the way Pachymeres organises the seriality of his problems: to one generic enunciation correspond five types of instantiation (according to the involved ratio), each type represented by a sub-series of concrete problems.

Let us see another example. The third problem of Diophantus asks to divide a given number into two numbers in a given ratio and difference (Tannery 1893–95, i, 18.8–9) – this is a technical expression meaning that one number is a given amount more than a given ratio of the other. In the instantiated version the given number is 80, the difference is 4, and the ratio is a triple (Tannery 1893–95, i, 18.10–11). Having this problem as a model Pachymeres creates 14 problems that he tackles in three chapters (33–35), as shown in Table 4.

Table 4.

Chapter	Kind of ratio	Number of problems
33	Multiple	5
34	Superparticular	5
35	Superpartient	4
	Multiple superparticular	
	Multiple superpartient	
		14

Table 5.

Chapter	Kind of ratio involved	Number of problems
26	No ratio involved (Dioph. I.1)	2
27–32	<i>To divide a given number into two numbers having a given ratio</i> (Dioph. I.2)	
27	Multiple	3
28	Superparticular	6
29	Superpartient	3
30–31	Multiple superparticular	8
32	Multiple superpartient	5
33–35	<i>To divide a given number into two such that the one be a given amount more than a ratio of the other</i> (Dioph. I.3)	
33	Multiple	5
34	Superparticular	5
35	Superpartient, Multiple superparticular, Multiple superpartient	4
36–38	<i>To divide a given number into two such that the one, if augmented by a given amount to form with the other a given ratio</i> ⁴	
36	Multiple	4
37	Superparticular	2
38	Superpartient, Multiple superparticular, Multiple superpartient	6
39	<i>To find two numbers in a given ratio and such that their difference is also given</i> (Dioph. I.4)	
39	Multiple	6
40	No ratio involved (Dioph. I.5–6)	5
41	<i>To two given numbers to add the same (required) number so as to make the resulting numbers have to one another a given ratio</i> (Dioph. I.8)	
41	Multiple	11
42–43	<i>From two given numbers to subtract the same (required) number so as to make the remainders have to one another a given ratio</i> (Dioph. I.9)	
42–43	Multiple	1
44a	<i>Given two numbers, to add to the lesser and to subtract from the greater the same (required) number so as to make the sum in the first case have to the difference in the second case a given ratio</i> (Dioph. I.10)	
44	Multiple	1
44b	<i>Given two numbers, to add the first to, and subtract the second from, the same (required) number, so as to make the resulting numbers have to one another a given ratio</i> (Dioph. I.11)	
44	Multiple	1

Again the generation of problems adheres to the same pattern as in problem I.2. The kind of the involved ratio is the criterion for the emergence of the three sub-series of problems.

On the whole, the series and sub-series of problems that Pachymeres creates out of the eleven problems of Diophantus are shown in Table 5.

⁴ This type of problem is not found in Diophantus.

5. Table 5 above illustrates how Pachymeres combines his two sources, Diophantus' *Arithmetica* and Nicomachus' *Arithmetical introduction*, in order to create a long series of arithmetical problems. It also displays the highly systematic manner with which he has arranged all these problems. Yet, creation and arrangement both refer to the enunciations of the problems; not to the solutions. Indeed, so far *the enunciations of the problems* have been the object of our investigation. Thus, the typology shown in Table 5 is in fact a typology of enunciations. As said, even at this level, the *savoir-faire* of the author is evident. However, Pachymeres' creative thinking is even more manifest in the sophisticated manner with which he employs the Nicomachean theory of ratios in *the solutions of the problems*. For, although all his problems stem ultimately from Diophantus, the solutions of Diophantus are only described in the last two problems, and even then they are given as second solutions,⁵ the first being, in both cases, solutions by the method of trial and error. In no other problem the method of Diophantus is employed in the solution, a fact which by itself confirms Hunger's claim, previously quoted in this article, that Pachymeres does not imitate blindly his sources. On the contrary, the way he particularly deals with the problems whose enunciations involve ratio, reveals not only an independent and unconventional spirit but also a talented thinker in mathematics.⁶

We shall conclude therefore this article by briefly discussing this aspect of Pachymeres' series of problems. To this end, let us see how Pachymeres employs the Nicomachean theory of ratios to the solution of a simple "Diophantine" problem, namely the second problem of the first book of the *Arithmetica*, which asks to divide a given number into two numbers forming a given ratio. In the instantiated version of the problem that Diophantus solves the ratio is of the kind of the multiple (namely, triple). As shown in Table 5 Pachymeres deals with problems of this type in chapters 27–32 of his arithmetic book of the *Quadrivium*, the case of the multiple ratio, in particular, being discussed in chapter 27.

Below we reproduce the introductory sentence and the first of the three problems of chapter 27:

"Well, the arithmetical problems in multiplications are as follows: if we are asked to divide the given number in ratio either double, or triple, or whatever, such that the part has to the part this ratio.

So, if we are asked to divide in double ratio, we must take the sub-triple of the whole, and make it the lesser term, of which we take the double, that is, the remainder, <which is> the greater term. And the problem is made.

For example, if we are asked to divide the number 24 into <two numbers having> double ratio, we are looking for its sub-triple, and it is 8. Its double, that is the remainder, 16, becomes the greater term, and the problem is satisfied. Indeed, 16 is the double of 8, and 16 and 8 <make> 24" (50.12–21).

The solution procedure can be represented by the following diagram (Table 6):

Table 6. Givens: 24 (the number to be divided), double ratio.

⁵ Both solutions are introduced by Pachymeres as "demonstrations" (ἀποδείξεις). Thus, in the first he writes "the demonstration of the preceding problem, according to the Alexandrian Diophantus, is the following" (75.17–18), while the second is preceded by the phrase "Therefore, in this manner, too, the preceding problem is demonstrated; while the second, <is demonstrated> as follows" (76.3–4).

⁶ Despite this, the old view expressed by Tannery, which sees the "Diophantine" part of the book on arithmetic as a mere "paraphrasis" of Diophantus is still repeated in some modern works on history of mathematics.

where the italicised numbers are those the finding of which the enunciation of the problem calls for. Apart from this solution Pachymeres proposes an alternative one, in which the second step instead of doubling involves subtraction (Table 7):

Table 7.

$$24 \xrightarrow{\text{sub-triple}} 8 \xrightarrow{\text{remainder } [24-8]} 16.$$

This variation put aside, both solutions rely on the following two presuppositions:

- (1) Since the given ratio is *double* one has to know that he must work with the *sub-triple* ratio, in order to carry through the first step of the procedure.
- (2) Knowing the number 24, he must find what number forms with it a pair in sub-triple ratio. That is, he must find the number 8, since the pair (8, 24) is in sub-triple ratio.

Of the above two prerequisites the former depends on the type of the problem. The latter, however, is not related to the problem. It is a kind of knowledge that the practitioner must have in advance. Thus, to solve problems of the type discussed here by the method taught by Pachymeres the practitioner must be able to answer questions regarding ratios of this type:

- (Q) Knowing one of the two terms of a pair (A, B) in a specific ratio, to find the other term of the pair.

This question is implied in all problems involving ratio that Pachymeres solves. And indeed its answer is not always as trivial as in problem 27.1. This is easily seen in the inventory presented in the appendix of this paper, where all instances from Pachymeres' solutions in chapters 27–38 are listed.

The inventory demonstrates that answering the question Q is not always an easy task. After all, it is clear that knowing merely the definitions of the several kinds of ratio is not enough. The question therefore that arises at this point is as simple as possible: is there any clue in Greek arithmetical literature where one would resort to for answering questions of this type? An answer might be that the “theory” of *pêlikotês* of numerical ratios, as we found it applied by Theon for example in his commentary to Ptolemy's *Almagest* and by Eutocius in his commentary to Archimedes' *On the Sphere and the Cylinder*, provides the tools for tackling questions of this type. Apart from the ‘theory’ of *pêlikotês* we claim now that the *Arithmetical introduction* provides an alternative and indeed easier way to do this. In a recent paper we have argued that the book of Nicomachus provides us with all the necessary instructions for answering such questions without resorting to the cumbersome calculations with fractional numbers that the way via *pêlikotês* often involves (Megremi & Christianidis 2014, 53–85). More specifically we claim that the arithmetical theory of ratios, as it is exposed in the treatise of Nicomachus, could be viewed in the context of a certain arithmetical practice, and that Pachymeres' solutions to the Diophantine problems of his *Quadrivium* provides us with a unique, within the Greek arithmetical tradition, case study of this practicality. The paper points out several features of Nicomachus' text, such as the many numerical tables it contains, the exhaustive instructions it gives on how the tables are constructed, the detailed explanations on how pairs of numbers that are in one or another ratio could be located within the tables, and it argues that these characteristics can be explained by assuming that a kind of arithmetical practice is intended. Answering the question above would be related to this practice. But the limited space of the present article prevents us from restating the arguments we develop there.

We would like to express our sincerest appreciation to Bernard Vitrac, Caroline Macé, Jeffrey Oaks and Alain Bernard, who read and commented on earlier versions of this paper.

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Appendix

Table showing the types of ratio and the terms of each ratio that are involved in the problems discussed in chapters 27–38 of the book on arithmetic of Pachymeres' *Quadrivium*.

Chapter	Problem	Given term (Antecedent)	Type of ratio	Term to be found (Consequent)
27	1	24	sub-triple	8
27	2	60	sub-quadruple	15
27	3	30	sub-quintuple	6
28	1	20	sub-double-sesquialter	8
28	2	30	sub-double-sesquialter	12
28	3	28	sub-double-sesquitercian	12
28	4	36	sub-double-sesquiquartan	16
28	5	22	sub-double-sesquiquintan	10
28	6	26	sub-double-sesquisextan	12
29	1	32	sub-double-superbitertient	12
29	2	22	sub-double-supertriartan	8
29	3	70	sub-double-superquadrartan	25
30	1	14	sub-triple-sesquialter	4
30	2	28	sub-triple-sesquialter	8
30	3	56	sub-triple-sesquialter	16
30	4	30	sub-triple-sesquitercian	9
30	5	26	sub-triple-sesquiquartan	8
31	1	18	sub-quadruple-sesquialter	4
31	2	39	sub-quadruple-sesquitercian	9
31	3	17	sub-quadruple-sesquiquartan	4
32	1	33	sub-triple-superbitertient	9
32	2	42	sub-quadruple-superbitertient	9
32	3	15	sub-triple-supertriartan	4
32	4	19	sub-triple-superquadrartan	5
32	5	23	sub-triple-superquintisextan	6
33	1	24	sub-triple	8
33	2	76	sub-quadruple	19
33	3	40	sub-quadruple	10
33	4	50	sub-quintuple	10
33	5	60	sub-sextuple	10
34	1	15	sub-double-sesquialter	6
34	2	20	sub-double-sesquialter	8
34	3	25	sub-double-sesquialter	10
34	4	21	sub-double-sesquitercian	9
34	5	18	sub-double-sesquiquartan	8
35	1	16	sub-double-superbitertient	6
35	2	14	sub-triple-sesquialter	4
35	3	24	sub-double-superbitertient	9
35	4	11	sub-double-supertriartan	4
36	1	30	sub-triple	10

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Chapter	Problem	Given term (Antecedent)	Type of ratio	Term to be found (Consequent)
36	2	31	sub-triple	$10 \frac{1}{3}$
36	3	32	sub-quadruple	8
36	4	50	sub-quintuple	10
37	1	20	sub-double-sesquialter	8
37	2	28	sub-double-sesquitercian	12
38	1	24	sub-double-superbitertient	9
38	2	33	sub-double-supertriquartan	12
38	3	14	sub-triple-sesquialter	4
38	4	27	sub-quadruple-sesquialter	6
38	5	26	sub-quadruple-sesquitercian	6
38	6	22	sub-triple-superbitertient	6

There follow 20 more ratios, included in chapters 39, 41, 42, 43, and 44, which are all instances of the kind of sub-multiple.