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Some remarks on the Hausdorff measure of the Cantor set

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ABSTRACT: In this paper, the author further reveals some intrinsic properties of the Cantor set. By erties, the author gives a new method for calculating the exact value of the Hausdorff measure of the C and shows the facts that each covering which consists of basic intervals, in which any basic interval can pletely contain the other, is the best covering of the Cantor set, and the Hausdorff measure of the Cant be determined by coverings which only consist of basic intervals.

Keywords: fractal; Hausdorff measure; Cantor set

1 INTRODUCTION

It is well known that it is one of the most important subjects to calculate or estimate Hausdorff measures of fractal sets in fractal geometry. Generally speaking, it is very difficult to calculate Hausdorff measures of fractal sets, as Falconer said in [1]: "It is already becoming apparent that calculation of Hausdorff measures and dimensions can be a little involved, even for simple sets. Usually it is the lower estimate that is awkward to obtain." So far, there are few concrete results about computation of Hausdorff measures even for some simple fractals. The Hausdorff measure of the Cantor set which equals 1 is shown in [2]. In [3], Ayer and Strichartz gave an algorithm for computing the Hausdorff measures of a class of Cantor sets. With the help of the principle of mass distribution, the Hausdorff measure of a Sierpinski carpet which equals $\sqrt{2}$ is obtained in [4]. The references [5] and [6] respectively gave the upper and lower bound for the Hausdorff measure of the Sierpinski gasket.

As what the authors pointed out in [7], the reason why it is so difficult to calculate the Hausdorff measure is neither computational trickiness nor computational capacity, but a lack of full understanding of the essence of the Hausdorff measure and a deep insight for the intrinsic properties of fractal sets. In this paper, we will further reveal some intrinsic properties of the Cantor set about Hausdorff measure. We show that each covering that consists of basic intervals in which any basic interval cannot completely contain the other is a best covering, and the Hausdorff measure of the Cantor set can be determined by coverings which only consist of basic intervals. It follows that a new method calculating the Hausdorff measure of the Cantor set is given.

2 PRELIMINARIES

Let U be a non-empty subset of R^n, which is denoted by $|U|$, the diameter of U. If $E \subset \bigcup_i U_i$ and $0 < |U_i| \le \delta$ for each i, we say that $\{U_i\}$ is a δ-covering of E. Let E be a subset of R^n and s be a non-negative number, for $\delta > 0$, it is defined as follows:

$$H_\delta^s(E) = \inf \sum_{i=1}^{\infty} |U_i|^s$$

Where, the infimum is over all (countable) δ-coverings $\{U_i\}$ of E. The Hausdorff s-dimensional measure of E is defined as follows:

$$H^s(E) = \lim_{\delta \to 0} H_\delta^s(E)$$

A class N of sets is called a net for E. If for any $x \in E$ and $\varepsilon > 0$, there exists $A \in N$ so that $x \in A$ and $|A| \le \varepsilon$. The s-dimensional net measure of E which is determined by the net N and denoted by

$H_N^s(E)$ is defined in a similar manner to the s-dimensional measure of E but using a restricted class N of covering sets in definition rather than the class of all sets. We say that the net N_1 is completely equivalent to the net N_2. If $H_{N_1}^s(E)=H_{N_2}^s(E)$ for any $E\subset R^n$ and any $s\geq 0$, the net N_1 is completely equivalent to the net N_2 for the set E and the number s if $H_{N_1}^s(E)=H_{N_2}^s(E)$. It is well known that the class of all sets is completely equivalent to the class of all the open sets and the class of all the closed sets.

Let $E_0=[0,1]$, $E_1=\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$, $E_3=\left[0,\frac{1}{9}\right]\cup\left[\frac{2}{9},\frac{1}{3}\right]\cup\left[\frac{2}{3},\frac{7}{9}\right]\cup\left[\frac{8}{9},1\right]$ and so on, where E_{k+1} is obtained by removing the middle third of each interval in E_k. Then E_k consists of 2^k intervals which are called the k th-stage basic intervals, and each of length is 3^{-k}. We can see that $E_0\supset E_1\supset E_2\supset\cdots\supset E_n\supset\cdots$. The Cantor set is the perfect set $C=\bigcap_{k=0}^{\infty}E_k$ that can be considered as a self-similarity set generated by two similitudes with the scale factor $1/3$. Since C satisfies the open set condition, the Hausdorff dimension of C equals the similarity dimension of C, $s_0=-\lg 2\Big/\lg\frac{1}{3}=\log_3^2$.

Let I_k be any k th-stage basic interval. We denote by F_k, the class of all the k th-stage basic intervals, and $F=\bigcup_{k=0}^{\infty}F_k$, the class of all the basic intervals. By the definition of a net, F is a net for C. Any two basic intervals of F are either disjoint or else one contained in the other.

3 SOME RESULTS

Since C can be covered by 2^k k th-stage basic intervals of length 3^{-k}, and by the definition of the Hausdorff s-dimensional measure, the following lemma holds obviously.

Lemma 1. $H^{s_0}(C)\leq 1$.

Lemma 2. Let I_k be any k th-stage basic interval, and $\{I_m\}(m>k)$ the class of all the m th-stage basic intervals contained in I_k. Then:

$$|I_k|^{s_0}=\sum_{I_m\in\{I_m\}}|I_m|^{s_0}.$$

Proof. Since I_k contains 2^{m-k} m th-stage basic intervals, each of length is 3^{-m}, and $|I_k|=3^{-k}, 3^{s_0}=2$, then $|I_k|^{s_0}=2^{-k}$, and:

$$\sum_{I_m\in\{I_m\}}|I_m|^{s_0}=2^{m-k}\left(3^{-m}\right)^{s_0}=2^{-k}$$

So the lemma holds.

Theorem 1. Let $\alpha=\{U_i\}$ be an arbitrary (countable) covering of C which consists of the basic intervals, and in which each of α cannot completely contain the other. Then:

$$\sum_{U_i\in\alpha}|U_i|^{s_0}=|E_0|^{s_0}=1$$

Proof. Let:

$$m=\inf\{j:U_i\in F_j\ \ for\ some\ U_i\in\alpha\}$$
$$n=\sup\{j:U_i\in F_j\ \ for\ some\ U_i\in\alpha\}$$

Then m is finite, and $n\geq m$ or $n=+\infty$. Next, we prove the theorem by three cases:

First, if $n=m$, then $\alpha=F_m$. It is shown as follows:

$$\sum_{U_i\in\alpha}|U_i|^{s_0}=\sum_{U_i\in F_m}|U_i|^{s_0}=2^m\left(3^{-m}\right)^{s_0}=1$$

Second, if $m<n<+\infty$, we suppose that there exist l_k k th-stage basic intervals, I_k in α, $k=m,m+1,\cdots,n$. Since each I_k contains 2^{n-k} n th-stage basic intervals, there are $l_m2^{n-m}+l_{m+1}2^{n-m-1}+\cdots+l_n$ n th-stage basic intervals in α. Since $\alpha=\{U_i\}$ is a covering of C, it is obvious that $l_m2^{n-m}+l_{m+1}2^{n-m-1}+\cdots+l_n=2^n$. By Lemma 2, it is shown as follows:

$$\sum_{U_i\in\alpha}|U_i|^{s_0}=l_m|I_m|^{s_0}+l_{m+1}|I_{m+1}|^{s_0}+\cdots+l_n|I_n|^{s_0}$$

$$=l_m2^{n-m}|I_n|^{s_0}+l_{m+1}2^{n-m-1}|I_n|^{s_0}+\cdots+l_n|I_n|^{s_0}$$

$$=\left(l_m2^{n-m}+l_{m+1}2^{n-m-1}+\cdots+l_n\right)|I_n|^{s_0}=2^n\left(3^{-n}\right)^{s_0}=1$$

Third, if $n=+\infty$, we suppose that there exist l_k k th-stage basic intervals, I_k in α, $k=m,m+1,\cdots$, then it is shown as follows:

$$\sum_{U_i\in\alpha}|U_i|^{s_0}=\sum_{k=m}^{+\infty}l_k|I_k|^{s_0}\tag{1}$$

Now we denote by m_p, the number of all the $(p+1)$ th-stage basic intervals, which are not contained in any q th-stage basic intervals in α where $q < p+1$. From the proof of the case $m < n < +\infty$, for any integer p $(p \geq m)$, we have as follows:

$$\sum_{k=m}^{p} l_k |I_k|^{s_0} + m_p |I_{p+1}|^{s_0} = 1 \qquad (2)$$

We claim that $m_p |I_{p+1}|^{s_0} \to 0$ as $p \to \infty$. In fact, if not, there exist $\varepsilon_0 > 0$ and positive integer $p_0 > N$ for any positive integer N so that $m_{p_0} |I_{p_0+1}|^{s_0} \geq \varepsilon_0$, and hence there exists an infinite subset N_{ε_0} of positive integers so that $m_p |I_{p+1}|^{s_0} \geq \varepsilon_0$ for $p \in N_{\varepsilon_0}$. For each $p \in N_{\varepsilon_0}$, let M_p be the union of m_p $(p+1)$ th-stage basic intervals, which is not contained in any q th-stage basic intervals in α where $q < p+1$. Obviously, M_p is non-empty and compact, and $M_j \supset M_i$ for $i > j$. Thus, $\bigcap_{p \in N_{\varepsilon_0}} M_p \neq \varnothing$ and $\bigcap_{p \in N_{\varepsilon_0}} M_p \subset C$. On one hand, $\varnothing \neq \bigcap_{p \in N_{\varepsilon_0}} M_p \subset \bigcup_{U_i \in \alpha} U_i$. On the other hand, from the definition of M_p, it follows that $\left(\bigcup_{U_i \in \alpha} U_i\right) \cap \left(\bigcap_{p \in N_{\varepsilon_0}} M_p\right) = \varnothing$, which is contrary to the fact that $\varnothing \neq \bigcap_{p \in N_{\varepsilon_0}} M_p \subset \bigcup_{U_i \in \alpha} U_i$. So we obtain from (2) that $\sum_{k=m}^{\infty} l_k |I_k|^{s_0} = 1$. Therefore, we obtain from (1) that $\sum_{U_i \in \alpha} |U_i|^{s_0} = 1$.

Finally, the theorem holds.

Corollary 1. $H_F^{s_0}(C) = |E_0|^{s_0} = 1$.

By the definition of the net measure and Theorem 1, it is obvious that the corollary holds.

Theorem 2. Let U be an open subset of R, and F_U be the family of the basic intervals I_σ contained completely in U, in which no one is contained in the other. Then it is shown as follows:

$$\sum_{I_\sigma \in F_U} |I_\sigma|^{s_0} \leq |U|^{s_0} \qquad (3)$$

Proof. By Theorem 1, it is obvious that $\sum_{I_\sigma \in F_U} |I_\sigma|^{s_0} \leq 1$. So the theorem holds when $|U| \geq 1$. Next, let $|U| < 1$, then there exists an integer $k \geq 0$ so that $3^{-(k+1)} \leq |U| < 3^{-k}$. It is obvious that U can only intersect with one k th-stage basic interval denoted by I and cannot completely contain I because of $|U| < 3^{-k}$. Since I contains two $(k+1)$ th-stage basic intervals, next, we prove the theorem by two cases.

First, if U intersects with one $(k+1)$ th-stage basic interval in I, then $\sum_{I_\sigma \in F_U} |I_\sigma|^{s_0} \leq |I_{k+1}|^{s_0} = \left(3^{-(k+1)}\right)^{s_0}$. Since $|U| \geq 3^{-(k+1)}$, it is shown that (3) holds.

Second, if U intersects with two $(k+1)$th-stage basic interval in I, we denote by $I_j^l (j > k)$ the j th-stage basic interval in I whose left end coincides with the left end of I, and by $I_j^r (j > k)$ the j th-stage basic interval in I whose right end coincides with the right end of I. Since $|U| = |\overline{U}| < 3^{-k}$, U and $\overline{U}$ cannot contain both the left end of I_{k+1}^l and the right end of I_{k+1}^r.

If either the left end of I_{k+1}^l or the right end of I_{k+1}^r is contained in $\overline{U}$, let us say that the left end of I_{k+1}^l is contained in $\overline{U}$, and the right end of I_{k+1}^r is not contained in $\overline{U}$, then there exists a positive integer t as follows:

$$U \cap I_{k+t}^r \neq \varnothing, \quad U \cap I_{k+t+1}^r = \varnothing$$

It is shown as follows:

$$|U| \geq |I| - |I_{k+t}^r| = 3^{-k} - 3^{-(k+t)} = \left(1 - \frac{1}{3^t}\right) 3^{-k} \qquad (4)$$

And:

$$\sum_{I_\sigma \in F_U} |I_\sigma|^{s_0} \leq |I_{k+1}^l|^{s_0} + (2^t - 1)|I_{k+t+1}|^{s_0} = \left(3^{-(k+1)}\right)^{s_0} + (2^t - 1)\left(3^{-(k+t+1)}\right)^{s_0}$$
$$= \left(\frac{1}{2} + \frac{2^t - 1}{2^{t+1}}\right)\left(3^{-k}\right)^{s_0} = \left(1 - \frac{1}{2^{t+1}}\right)\left(3^{-k}\right)^{s_0} \qquad (5)$$

We claim that:

$$\left(1 - \frac{1}{3^t}\right)^{s_0} \geq 1 - \frac{1}{2^{t+1}}, \text{ i.e. } \log_3^2 \cdot \ln\left(1 - \frac{1}{3^t}\right) \geq \ln\left(1 - \frac{1}{2^{t+1}}\right) \qquad (6)$$

In fact, it is obvious that (6) holds for $t = 1$. When $t \geq 2$, let $f(x) = \log_3^2 \cdot \ln\left(1 - \frac{1}{3^x}\right) - \ln\left(1 - \frac{1}{2^{x+1}}\right)$. We can see that $f(x)$ decreases on $[2, +\infty)$ by calculating the derivative $f'(x)$, and $f(+\infty) = \lim_{x \to +\infty} f(x) = 0$, and hence $f(x) \geq 0$ for $x \geq 2$. So we know from (4) and (5) that (3) holds.

If $\overline{U}$ can contain none of both the left end of

I_{k+1}^{l} and the right end of I_{k+1}^{r}, then there exist two positive integers m and n as follows:

$$U \cap I_{k+m}^{l} \neq \varnothing, U \cap I_{k+m+1}^{l} = \varnothing, U \cap I_{k+n}^{r} \neq \varnothing,$$
$$U \cap I_{k+n+1}^{r} = \varnothing$$

It follows that:

$$|U| \geq |I| - \left(\left|I_{k+m}^{l}\right| + \left|I_{k+n}^{r}\right|\right) = 3^{-k} - \left(3^{-(k+m)} + 3^{-(k+n)}\right) = \left(1 - \frac{1}{3^m} - \frac{1}{3^n}\right)3^{-k}$$

And:

$$\sum_{I_\sigma \in F_U} |I_\sigma|^{s_0} \leq \left(2^m - 1\right)\left|I_{k+m+1}^{l}\right|^{s_0} + \left(2^n - 1\right)\left|I_{k+n+1}\right|^{s_0}$$

$$= \left(2^m - 1\right)\left(3^{-(k+m+1)}\right)^{s_0} + \left(2^n - 1\right)\left(3^{-(k+n+1)}\right)^{s_0} = \left(1 - \frac{1}{2^{m+1}} - \frac{1}{2^{n+1}}\right)\left(3^{-k}\right)^{s_0}$$

Similar to the above discussion, we can get $\left(1 - \frac{1}{3^m} - \frac{1}{3^n}\right)^{s_0} \geq 1 - \frac{1}{2^{m+1}} - \frac{1}{2^{n+1}}$, and hence (3) holds.

Finally, the theorem holds.

Theorem 3. $H^{s_0}(C) = H_F^{s_0}(C) = 1$.

Proof. By Lemma 1 and Corollary 1, $H^{s_0}(C) \leq H_F^{s_0}(C)$ is obvious. To prove the opposite inequality, we need to consider all δ-coverings of C by the definition of $H^s(E)$. Since the class of all sets is completely equivalent to the class of all the open sets, it is enough to prove $H^{s_0}(C) \geq H_F^{s_0}(C)$ for all open δ-coverings of C. Now, let $\{U_i\}$ be an arbitrary open δ-covering of C. For any given $U_i \in \{U_i\}$, let F_{U_i} be the class of the basic intervals contained completely in U_i, in which no one is contained in the other, then $|U_i|^{s_0} \geq \sum_{I_\sigma \in F_{U_i}} |I_\sigma|^{s_0}$ according to Theorem 2. Since C is compact, we can make F_{U_i} satisfy $C \subset \bigcup_{U_i \in \{U_i\}} \bigcup_{I_\sigma \in F_{U_i}} I_\sigma$. Since $\sum_{U_i \in \{U_i\}} |U_i|^{s_0} \geq \sum_{U_i \in \{U_i\}} \sum_{I_\sigma \in F_{U_i}} |I_\sigma|^{s_0}$, $H^{s_0}(C) \geq H_F^{s_0}(C)$ holds. So we obtain $H^{s_0}(C) = H_F^{s_0}(C)$, and hence the theorem holds.

Corollary 2. For the Cantor set C and the number $s_0 = \log_3^2$, the class of all sets is completely equivalent to the net F which is the class consisting of all basic intervals.

In [7], the authors pose eight open problems on the exact value of the Hausdorff measure. The first open problem is shown as follows.

Problem 1. Under what condition is there a covering of E, say $\alpha = \{U_i, i > 0\}$, so that $H^s(E) = \sum_{i=1}^{m} |U_i|^s$?

Such a covering of E is called as a best covering.

It is easy to see that $\{C\}$ is a best covering of C, and that the class F_k of all the k th-stage basic intervals is the best covering of C.

By Theorem 1 and Theorem 3, we can obtain the following result.

Corollary 3. Let $\alpha = \{U_i\}$ be any covering of C consisting of basic intervals in which each of α cannot completely contain the other, then $\alpha = \{U_i\}$ is the best covering of C.

Note that $\alpha = \{U_i\}$ in Corollary 3 may be infinite, therefore, it is non-trivial.

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