Some remarks on the Hausdorff measure of the Cantor set

Minghua Wang
Chongqing University of Arts and Sciences, Yongchuan, Chongqing, China

ABSTRACT: In this paper, the author further reveals some intrinsic properties of the Cantor set. By properties, the author gives a new method for calculating the exact value of the Hausdorff measure of the Cantor set and shows the facts that each covering which consists of basic intervals, in which any basic interval can completely contain the other, is the best covering of the Cantor set, and the Hausdorff measure of the Cantor set can be determined by coverings which only consist of basic intervals.

Keywords: fractal; Hausdorff measure; Cantor set

1 INTRODUCTION

It is well known that it is one of the most important subjects to calculate or estimate Hausdorff measures of fractal sets in fractal geometry. Generally speaking, it is very difficult to calculate Hausdorff measures of fractal sets, as Falconer said in [1]: “It is already becoming apparent that calculation of Hausdorff measures and dimensions can be a little involved, even for simple sets. Usually it is the lower estimate that is awkward to obtain.” So far, there are few concrete results about computation of Hausdorff measures even for some simple fractals. The Hausdorff measure of the Cantor set which equals 1 is shown in [2]. In [3], Ayer and Strichartz gave an algorithm for computing the Hausdorff measures of a class of Cantor sets. With the help of the principle of mass distribution, the Hausdorff measure of a Sierpinski carpet which equals 2 is obtained in [4]. The references [5] and [6] respectively gave the upper and lower bound for the Hausdorff measure of the Sierpinski gasket.

As what the authors pointed out in [7], the reason why it is so difficult to calculate the Hausdorff measure is neither computational trickiness nor computational capacity, but a lack of full understanding of the essence of the Hausdorff measure and a deep insight for the intrinsic properties of fractal sets. In this paper, we will further reveal some intrinsic properties of the Cantor set about Hausdorff measure. We show that each covering that consists of basic intervals in which any basic interval cannot completely contain the other is a best covering, and the Hausdorff measure of the Cantor set can be determined by coverings which only consist of basic intervals. It follows that a new method calculating the Hausdorff measure of the Cantor set is given.

2 PRELIMINARIES

Let $U$ be a non-empty subset of $\mathbb{R}^n$, which is denoted by $U$, the diameter of $U$. If $E \subset \bigcup U_i$ and $0 < |U_i| \leq \delta$ for each $i$, we say that $\{U_i\}$ is a $\delta$-covering of $E$. Let $E$ be a subset of $\mathbb{R}^n$ and $s$ be a non-negative number, for $\delta > 0$, it is defined as follows:

$$H_s^\delta (E) = \inf \sum_{i=1}^\infty |U_i|$$

Where, the infimum is over all (countable) $\delta$-coverings $\{U_i\}$ of $E$. The Hausdorff $s$-dimensional measure of $E$ is defined as follows:

$$H_s^\delta (E) = \lim_{\delta \to 0} H_s^\delta (E)$$

A class $N$ of sets is called a net for $E$. If for any $x \in E$ and $\varepsilon > 0$, there exists $A \in N$ so that $x \in A$ and $|A| \leq \varepsilon$. The $s$-dimensional net measure of $E$ which is determined by the net $N$ and denoted by

This is an Open Access article distributed under the terms of the Creative Commons Attribution License 4.0, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
$H_{\beta}(E)$ is defined in a similar manner to the $s$-dimensional measure of $E$ but using a restricted class $N'$ of covering sets in definition rather than the class of all sets. We say that the net $N_i$ is completely equivalent to the net $N_j$. If $H_{\beta_i}(E) = H_{\beta_j}(E)$ for any $E \subset \mathbb{R}^n$ and any $\beta \geq 0$, the net $N_i$ is completely equivalent to the set $E$ and the number $\beta$ if $H_{\beta_i}(E) = H_{\beta_j}(E)$. It is well known that the class of all sets is completely equivalent to the class of all the open sets and the class of all the closed sets.

Let $E_0 = [0, 1]$, $E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ and so on, where $E_{k+1}$ is obtained by removing the middle third of each interval in $E_k$. Then $E_k$ consists of $2^k$ intervals which are called the $k$th basic intervals, and each of length is $3^{-k}$. We can see that $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$, and the Cantor set is the perfect set $C = \bigcap_{k=0}^{\infty} E_k$ that can be considered as a self-similarity set generated by two similitudes with the scale factor $1/3$. Since $C$ satisfies the open set condition, the Hausdorff dimension of $C$ equals the similarity dimension of $C$, $s_0 = -\log_{3/2} 1 = \log_3 2$.

Let $I_k$ be any $k$th basic interval. We denote by $F_k$, the class of all the $k$th basic intervals, and $F = \bigcup_{k=0}^\infty F_k$, the class of all the basic intervals. By the definition of a net, $F$ is a net for $C$. Any two basic intervals of $F$ are either disjoint or else one contained in the other.

### 3 SOME RESULTS

Since $C$ can be covered by $2^k$ $k$th basic intervals of length $3^{-k}$, and by the definition of the Hausdorff $s$-dimensional measure, the following lemma holds obviously.

**Lemma 1.** $H^s(C) \leq 1$.

**Lemma 2.** Let $I_k$ be any $k$th basic interval, and $|I_k|$ $(m > k)$ the class of all the $m$ th-stage basic intervals contained in $I_k$. Then:

$$|I_k|^s = \sum_{k=0}^{m} |I_k|^s.$$

**Proof.** Since $I_k$ contains $2^{m-k}$ $m$th basic intervals, each of length is $3^{-m}$, and $|I_k| = 3^{-k}3^m = 2$, then $|I_k|^s = 2^{-k}$, and:

$$\sum_{k=0}^{m} |I_k|^s = 2^{-k}(3^{-m})^s = 2^{-k}$$

So the lemma holds.

**Theorem 1.** Let $\alpha = \{U_i\}$ be an arbitrary (countable) covering of $C$ which consists of the basic intervals, and in which each of $\alpha$ cannot completely contain the other. Then:

$$\sum_{i=0}^{n} |U_i|^s = |E_0|^s = 1$$

**Proof.** Let:

$$m = \inf \left\{ j : U_i \in F_j \text{ for some } U_i \in \alpha \right\}$$

$$n = \sup \left\{ j : U_i \in F_j \text{ for some } U_i \in \alpha \right\}$$

Then $m$ is finite, and $n \geq m$ or $n = +\infty$. Next, we prove the theorem by three cases:

First, if $n = m$, then $\alpha = F_m$. It is shown as follows:

$$\sum_{i=0}^{n} |U_i|^s = \sum_{i=0}^{n} |U_i|^s = 2^n(3^{-m})^s = 1$$

Second, if $m < n < +\infty$, we suppose that there exist $l_k$ $k$th basic intervals, $I_k$ in $\alpha$, $k = m, m+1, \cdots$. Since each $I_k$ contains $2^{m-k}$ $n$th basic intervals, there are $l_k 2^{m-k} + l_{m+1}2^{m-k-1} + \cdots + l_n$ th-stage basic intervals in $\alpha$. Since $\alpha = \{U_i\}$ is a covering of $C$, it is obvious that $I_k 2^{m-n} + l_{m+1}2^{m-n-1} + \cdots + l_n = 2^n$. By Lemma 2, it is shown as follows:

$$\sum_{i=0}^{n} |U_i|^s = \sum_{i=0}^{n} |U_i|^s + l_{m+1}2^{m-k} |I_k|^s + \cdots + l_n |I_k|^s$$

$$= l_m 2^{m-n} |I_k|^s + l_{m+1}2^{m-n} |U_i|^s + \cdots + l_n |I_k|^s$$

$$= (l_m 2^{m-n} + l_{m+1}2^{m-n-1} + \cdots + l_n) |I_k|^s = 2^n(3^{-m})^s = 1$$

Third, if $n = +\infty$, we suppose that there exist $l_k$ $k$th basic intervals, $I_k$ in $\alpha$, $k = m, m+1, \cdots$, then it is shown as follows:

$$\sum_{i=0}^{n} |U_i|^s = \sum_{k=m}^{\infty} |I_k|^s$$

(1)

Now we denote by $m_p$, the number of all the $(p+1)$th basic intervals, which are not con-
tained in any $q$ th-stage basic intervals in $\alpha$ where $q < p + 1$. From the proof of the case $m < n < +\infty$, for any integer $p (p \geq m)$, we have as follows:

$$\sum_{k=1}^{n} |I_k|^s + m_p |I_{p+1}|^s = 1 \quad (2)$$

We claim that $m_p |I_{p+1}|^s \to 0$ as $p \to \infty$. In fact, if not, there exist $E_0 > 0$ and positive integer $p_0 > N$ for any positive integer $N$ so that $m_p |I_{n+1}|^s \geq E_0$, and hence there exists an infinite subset $N_{e_0}$ of positive integers so that $m_p |I_{p+1}|^s \geq E_0$ for $p \in N_{e_0}$. For each $p \in N_{e_0}$, let $M_p$ be the union of $m_p (p+1)$ th-stage basic intervals, which is not contained in any $q$ th-stage basic intervals in $\alpha$ where $q < p + 1$. Obviously, $M_p$ is non-empty and compact, and $M_p \supseteq M_i$ for $i > j$. Thus, $\cap_{p \in N_{e_0}} M_p \neq \emptyset$ and $\cap_{p \in N_{e_0}} M_i = C$. On one hand, $\emptyset \neq \cap_{p \in N_{e_0}} M_p \subset \cup_{i \in \mathbb{N}} U_i$. On the other hand, from the definition of $M_p$, it follows that $\cup_{i \in \mathbb{N}} U_i \cap \left( \cap_{p \in N_{e_0}} M_p \right) = \emptyset$ which is contrary to the fact that $\emptyset \neq \cap_{p \in N_{e_0}} M_p \subset \cup_{i \in \mathbb{N}} U_i$. So we obtain from (2) that

$$\sum_{p \in N_{e_0}} |I_p|^s = 1.$$  

Therefore, we obtain from (1) that

$$\sum_{k=1}^{n} |I_k|^s = 1.$$  

Finally, the theorem holds.

**Corollary 1.** $H^{s} \left( C \right) = \left| E_0 \right|^s = 1.$  

By the definition of the net measure and Theorem 1, it is obvious that the corollary holds.

**Theorem 2.** Let $U$ be an open subset of $\mathbb{R}$, and $F_U$ be the family of the basic intervals $I_\alpha$ contained completely in $U$, in which no one is contained in the other. Then it is shown as follows:

$$\sum_{\alpha \in F_U} |I_\alpha|^s \leq \left| I \right|^s.$$  

**Proof.** By Theorem 1, it is obvious that $\sum_{\alpha \in F_U} |I_\alpha|^s \leq 1$. So the theorem holds when $|I| \geq 1$. Next, let $|I| < 1$, then there exists an integer $k \geq 0$ so that $\left( 3^{-k} \right) < |I| < 3^{-k}$. It is obvious that $U$ can only intersect with one $\bar{k}$ th-stage basic interval denoted by $I$ and cannot completely contain $I$ because of $|I| < 3^{-k}$. Since $I$ contains two $(k + 1)$ th-stage basic intervals, next, we prove the theorem by two cases.

First, if $U$ intersects with one $(k + 1)$ th-stage basic interval in $I$, then

$$\sum_{\alpha \in F_U} |I_\alpha|^s \leq |I_{\alpha+1}|^s = \left( 3^{-(k+1)} \right)^s.$$  

Since $|I| \geq \left| I \right|^s$, it is shown that (3) holds.

Second, if $U$ intersects with two $(k + 1)$ th-stage basic interval in $I$, we denote by $I_{\alpha+1}(j > k)$ the $j$ th-stage basic interval in $I$ whose left end coincides with the left end of $I$, and by $I_{\alpha+1}(j > k)$ the $j$ th-stage basic interval in $I$ whose right end coincides with the right end of $I$. Since $\left| I \right| = |\left| U \right| < 3^{-k}$, $U$ and $\mathcal{U}$ cannot contain both the left end of $I_{\alpha+1}$ and the right end of $I_{\alpha+1}$.

If either the left end of $I_{\alpha+1}$ or the right end of $I_{\alpha+1}$ is contained in $\mathcal{U}$, let us say that the left end of $I_{\alpha+1}$ is contained in $\mathcal{U}$, and the right end of $I_{\alpha+1}$ is not contained in $\mathcal{U}$, then there exists a positive integer $t$ as follows:

$$U \cap I_{\alpha+1} \neq \emptyset, \quad U \cap I_{\alpha+1} = \emptyset$$

It is shown as follows:

$$|I| \geq \left| I_{\alpha+1} \right|^s = 3^{-3} - 3^{-3(t+1)} = \left( 1 - \frac{1}{3} \right)^3 = \left( 1 - \frac{1}{3} \right)^3 \quad (4)$$

And:

$$\sum_{\alpha \in F_U} |I_\alpha|^s \leq \left| I_{\alpha+1} \right|^s + \left( 3^{-3} - 3^{-3(t+1)} \right) = \left( \frac{1}{2} + \frac{3}{2} \right) \left( 3^{-3} \right)^s = \left( 1 - \frac{1}{2} \right) \left( 3^{-3} \right)^s \quad (5)$$

We claim that:

$$\left( 1 - \frac{1}{2} \right) \left( 3^{-3} \right)^s \geq 1 - \frac{1}{2^n}, \text{ i.e. } \log_3 \left( 1 - \frac{1}{2^n} \right) \geq \log_3 \left( 1 - \frac{1}{2^n} \right) \quad (6)$$

In fact, it is obvious that (6) holds for $t = 1$. When $t \geq 2$, let $f(x) = \log_3 \left( 1 - \frac{1}{2^n} \right)$, we can see that $f(x)$ decreases on $[2, +\infty)$ by calculating the derivative $f'(x)$, and $f(\infty) = \lim_{x \to \infty} f(x) = 0$, and hence $f(x) \geq 0$ for $x \geq 2$. So we know from (4) and (5) that (3) holds.

If $\mathcal{U}$ can contain none of both the left end of
$I'_{k+1}$ and the right end of $I'_{k+1}$, then there exist two positive integers $m$ and $n$ as follows:

$U \cap I'_{k+1} \neq \emptyset, U \cap I'_{k+1} = \emptyset, U \cap I'_{k+1} \neq \emptyset,$

$U \cap I'_{k+1} = \emptyset$

It follows that:

$|F[|I'|_{k+1} + |I'|_{k+1} + |I'|_{k+1}| = 3 \cdot \left(3^{(\alpha_{k+1})} + 3^{(\alpha_{k+1})}\right) = \left(1 - \frac{1}{3^2} - \frac{1}{3^2}\right)3^{(\alpha_{k+1})}$

And:

$\sum_{i \in I_{k}} |I_{k+1}^i| \geq (2^m - 1) |I_{k+1}^i| + (2^n - 1) |I_{k+1}^n|$

$= (2^m - 1) (3^{(\alpha_{k+1})}) + (2^n - 1) (3^{(\alpha_{k+1})}) = \left(1 - \frac{1}{2^{m}} - \frac{1}{2^{n}}\right)3^{(\alpha_{k+1})}$

Similar to the above discussion, we can get (3) holds.

Finally, the theorem holds.

**Theorem 3.** $H_{\alpha}^{\gamma}(C) = H_{\alpha}^{\gamma}(C) = 1.$

**Proof.** By Lemma 1 and Corollary 1, $H_{\alpha}^{\gamma}(C) \leq H_{\beta}^{\gamma}(C)$ is obvious. To prove the opposite inequality, we need to consider all $\delta$-coverings of $C$ by the definition of $H_{\alpha}^{\gamma}(E)$. Since the class of all sets is completely equivalent to the class of all open sets, it is enough to prove $H_{\alpha}^{\gamma}(C) \geq H_{\alpha}^{\gamma}(C)$ for all open $\delta$-coverings of $C$. Now, let $\{U_i\}$ be an arbitrary open $\delta$-covering of $C$. For any given $U_i \in \{U_i\}$, let $F_{U_i}$ be the class of the basic intervals contained completely in $U_i$, in which no one is contained in the other, then $|V| = \sum_{i \in V} |I_i|$ according to Theorem 2. Since $C$ is compact, we can make $F_{U_i}$ satisfy $C \subseteq \bigcup_{i \in V} I_i$. Since $\sum_{i \in V} |V| \geq \sum_{i \in V} |I_i|$, $H_{\alpha}^{\gamma}(C) \geq H_{\alpha}^{\gamma}(C)$ holds. So we obtain $H_{\alpha}^{\gamma}(C) = H_{\alpha}^{\gamma}(C)$, and hence the theorem holds.

**Corollary 2.** For the Cantor set $C$ and the number $s = \log_3 \frac{1}{2}$, the class of all sets is completely equivalent to the net $C$ which is the class consisting of all basic intervals.

In [7], the authors pose eight open problems on the exact value of the Hausdorff measure. The first open problem is shown as follows.

**Problem 1.** Under what conditions is there a covering of $E$, say $\alpha = \{U_i, i > 0\}$, so that

$H_{\alpha}^{\gamma}(E) = \sum_{i \in \alpha} |U_i|$?

Such a covering of $E$ is called as a best covering.

It is easy to see that $\{C\}$ is a best covering of $C$, and that the class $F_k$ of all the $k$-th-stage basic intervals is the best covering of $C$.

By Theorem 1 and Theorem 3, we can obtain the following result.

**Corollary 3.** Let $\alpha = \{U_i\}$ be any covering of $C$ consisting of basic intervals in which each of $\alpha$ cannot completely contain the other, then $\alpha = \{U_i\}$ is the best covering of $C$.

Note that $\alpha = \{U_i\}$ in Corollary 3 may be infinite, therefore, it is non-trivial.

**REFERENCES**


