Mathematical Model with Feedback Loop in Company Control

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Abstract. This article is devoted to the further development of feedback loop models. They are used in the management of a single-industry firm. The development of the studying consists in the mathematical modeling of transients and assessing their impact on economic indicators. To solve this problem, the following was done in the article. A structural model of the functioning of a single-industry firm (SIF) is presented. It is built in accordance with the theory of automatic control (TAC) and includes elements: a management unit, a production unit, a sales unit, information nodes, as well as a feedback loop (FL). Equations and relationships describing the logic of the functioning of the company as a production system are presented. They allowed us to derive dynamic relationships and differential equations that reflect feedback loops on revenue and production costs. The system of expressions in operator form is presented, which describes the contour of the FL SIF. It has the form of a system of differential equations. It forms the basis of the mathematical model of SIF in the control system. This model made it possible to obtain a graphical interpretation of transients with closed and open FL based on the use of the Mathcad editor. Transients are fluctuations in sales volumes and production costs in the presence of disturbing influences. This is the scientific result and determines the novelty of the article.

1 Introduction

The presence of feedback loop in the management system is a prerequisite for ensuring the effectiveness and sustainability of the functioning and development of the company. It is difficult to monitor the compliance of actually achieved parameters of functioning and development with the given ones without fulfilling this condition. It is impossible to make managerial decisions on the adjustment of strategy elements, planned indicators, timelines, methods and means of management, etc. Therefore, the feedback loop principle has a certain distribution in the mathematical modeling of economic objects and aspects of their activities. However, the analysis of scientific works on this issue showed: a mathematical model was not built that allows us to interpret the nature of transients. Fluctuations in sales volumes and production costs under disturbing influences are considered as transients with a closed and open feedback loop. Such an interpretation is of both scientific and practical interest. It

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contributes to the further development of the methodology of mathematical modeling in the management of economic systems. It is aimed at increasing management efficiency by building a stable feedback loop. The feedback loop provides the necessary operational information for making and monitoring the implementation of management decisions. Thus, the purpose of the article is to build a mathematical model that allows you to perform a graphical interpretation of the nature of transients with a closed and open feedback loop.

2 Theoretical analysis

Mathematical modeling of the activity of economic systems and management systems is reflected in a number of studies. This formed an independent field both in mathematics and in management science. Of particular interest, according to the authors of the article, are models that take into account the influence of feedback, which is justified in the introduction. We will briefly review and analyze developments in this area. D. Forrester at work [1], examined the dynamic modeling of the company. He described the theory of feedback loop information system management. Introduced cybernetic models of dynamic systems with feedback. His research laid the foundation for systematic research in this area. One of the largest specialists in the field of TAC, academician V.A. Trapeznikov, in his work [2], considered the possibility of using theory in economics. The study was deployed in a book [3]. This approach in a generalized form was presented in the work [4]. It briefly analyzes new research in this direction. Proceedings of Sirazetdinova T.K. [5], Sio K.K. [6], Bonilla M. [7], Dooren V. [8], Klir G. [9], Checkland P. [10], Tolk Andreas [11], Lychkina N.N.[12], Slivitsky A.B. [13] devoted to various aspects of the construction and application of cybernetic and mathematical models based on the use of the feedback. These studies cover a wide range of issues and problems of the theory, methodology, methodology and practice of modeling. In particular, the principles, conditions and limitations of modeling, functioning, development and management of various economic systems are presented. Methodological foundations and methodological tools for constructing and using models with feedback loop are developed. A systematization is proposed, the possibilities and limitations of using models with feedback loop in the management of economic objects and their areas of activity are analyzed.

In addition to Forrester J, Jennings N R [14, 15], Wooldridge M.J [16] and others contributed to the formation of data of multi-agent resource conversion processes and dynamic liner modeling systems. Developments on distributed automated systems for optimizing enterprise processes are presented in the works of K.A. Aksenov with co-authors [17,18,19], as well as works [20, 21] on the development of data preparation modules in the framework of mathematical modeling of enterprise management systems, however, these works did not conduct a mathematical study of the effect of feedback loop on the parameters of the state and dynamics of economic systems. Structural schemes and models were built in articles [22-24]. Equations of the functioning of the enterprise as a production and economic system are also derived using the scientific apparatus of TAC. The possibilities of applying the proposed methodology to assess the sustainability of innovative development of enterprises in the innovation management system are substantiated. A number of points on the application of the TAC methodology in economic systems will be presented below. It should be noted that these schemes and models have received practical testing. This has resulted in the improvement of economic analysis and operational planning by the activity of a single-industry instrument-making enterprise (single-industry is an enterprise that produces homogeneous products). The proposed approach in the form of a virtual system for automated control of the activity of an instrument-making enterprise received practical implementation in the form of a patent of the Russian Federation. This proves the adequacy of the model.

In these studies, an extensive scientific apparatus and the practice of using models with feedback loop in the management of activities and the development of economic systems have been developed. But until now, a mathematical model has not been built that allows us to obtain a graphical interpretation of the nature of transients with a closed and open feedback loop. This is necessary to identify the impact of the feedback loop on the economic performance of the company. This determined the scientific task and focus of the research article.

3 Study results

None of the economic and mathematical models can fully reflect the complexity and diversity of the processes of functioning and development of the company. This is due to the difficult formalizability of the description and presentation of economic objects and processes in an optimal form for modeling. We offer some simplifications and assumptions. What is manifested in the following initial ideas.

1.   To solve this scientific problem, the article considers just a single-industry firm (a company that produces homogeneous products). Diversified production requires much more complex multi-level bulky models, which is extremely difficult and impractical. The structure and logic of the SIF functioning are presented in the form of a structural diagram (Fig. 1). Two feedback loops are marked on the diagram - by revenue and cost price (which distinguishes it from [25]), taking into account inertia (time delays). The structural model demonstrates the logic of the process of production `TAC.

2.   The planned value of the volume of output is introduced into the SIF activity model. The model includes blocks of production and sales of products with feedback loops. This approach allows you to evaluate deviations from the planned cost, production volume and sales, affecting the stability of the company. In the model SIF continuously produces and sells products. SIF has a steady volume of production and sales.

3.   It should be understood that in the exact sciences, methods are used to derive differential equations of motion of mechanical systems based on Lagrange equations of the 2nd kind, based on the kinetostatic method, Maxwell's equations, etc. The derivation of economic and mathematical models involves the use of structural analysis methods. Blocks have transfer functions, which are obtained by the experimental-statistical method. Based on the constructed block diagram, the transfer functions of the blocks are determined. The transfer functions and ratios for individual circuits consisting of blocks are derived. Next, two differential equations are derived in the time domain for cost (4th order equation) and revenue (1st order equation).

Under the transition process is understood the process of restoring the values (rates of change) of economic indicators after exposure to certain disturbances.
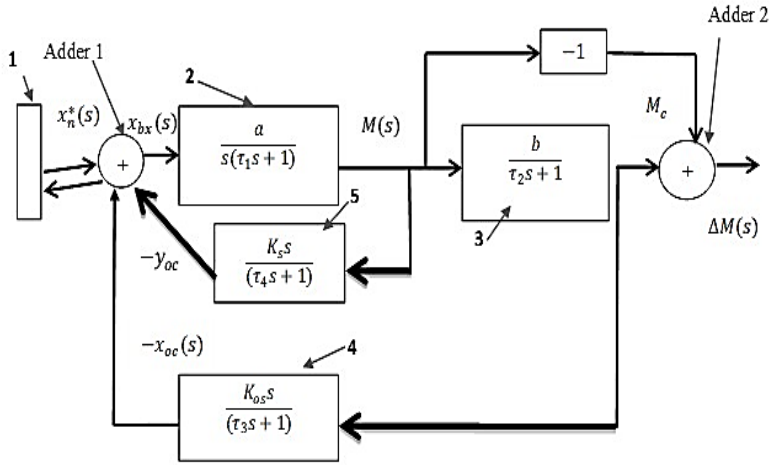
Fig. 1. Block diagram of SIF with feedback loops.

Designations: 1 - management unit; 2 - production unit; 3 - block sales of products; 4 - feedback loop on revenue from product sales; 5– cost feedback loop; summing blocks (in circuitry TAC-adders) 1 and 2 - information nodes in which the comparison of planned and actual indicators is carried out.

In the unit 1, a comparison is made of the rate of change in the planned volume of production $x_n^*(s)$ (in accordance with the notation system adopted in TAC - the Laplace image of the input impact on the system) with the actual. And accordingly, the planned and actual rate of sales, presented in the form of a derivative of revenue $M_c(s)$. The unit 1 generates input management actions on the system;

$M_n(s)$ - image of the planned cost of production;

K_{os} - feedback coefficient on revenue from sales of products, the value of which is derived from the ratio:

$$sK_{os}M_c(s) = x_{os}(s),$$

where $x_{os}(s)$ – the rate of change in the volume of sales;

K_s – feedback coefficient on the cost of production, entered in the transfer function (4a), where the product $K_s s$ appears;

$M(s)$ - Laplace image of the cost of production;

b – the level of profitability of the product;

$M_c(s)$ – image of the actual revenue of the enterprise (volume of sales);

$\Delta M(s)$ – profit from sales of products (Rub.);

τ_1, τ_2, τ_3, τ_4 - time constants characterizing the inertia (delay processes) of the corresponding blocks.

y_{oc} (s) – the feedback signal of the circuit at cost.

Let us derive the differential equations of the firm's functioning according to the structural diagram of Fig. 1 taking into account the time inertia (the presence of time lags M), which take place in real activity. Differential equations from [22], [23] are introduced. Additional members are introduced into them, taking into account the following representations.

For simplicity of further calculations, we introduce the notation:

$$\frac{K_s s}{\tau_4 s + 1} = K_\alpha. \tag{1}$$

- transfer function of the production unit: $\frac{a}{s(\tau_1 s+1)}$; (1.a)

- transfer function of the sales unit: $\frac{b}{\tau_2 s+1}$; (1.b)

- transfer function of the revenue feedback loop: $\frac{K_{os}s}{\tau_3 s+1}$; (1.c)

- the transfer function of the feedback loop at cost $\frac{K_s s}{\tau_4 s+1}$ (1.d)

After that we derive relations for interconnection of blocks.

For adder 1, we have the sum of the input variables

$$x_{\text{BX}}(s) = x_n^*(s) - x_{os}(s) - y_{os}(s) = x_n^*(s) - K_s M(s) - \frac{K_{os} s M_c(s)}{\tau_3 s+1}. \qquad (2)$$

For product sales unit 3, respectively:

$$M_c(s) = x_{\text{BX}}(s) * W_2(s) * W_3(s) \qquad . \qquad (3)$$

Where $W_2(s)$, $W_3(s)$ - transmission functions of production unit 2 and sales unit (realization) 3.

Substitute (5) in (6), get when taking into account expressions

$$M(s) = \frac{a x_{\text{BX}}(s)}{s(\tau_1 s+1)} \qquad , \qquad M_c(s) = \frac{b M(s)}{\tau_2 s+1}, \qquad (4)$$

$$M_c(s) = \frac{a}{s(\tau_1 s+1)} * \frac{b}{\tau_2 s+1} * \left(x_n^*(s) - K_s M(s) - \frac{K_{os} s M_c(s)}{\tau_3 s+1}\right). \qquad (5)$$

The resulting expressions describe the process of functioning of the company. They make it possible to obtain dynamic finite relations in the form of differential equations of the system for feedback options for revenue, cost and their totality.

The derivation of these differential equations implies obtaining a system of expressions in operator form:

$$W_{3s} = \frac{W(S)*W_n(S)}{1+W(S)*W_n(S)W_{os}(S)} , \qquad W_{os} = \frac{K_{os}S}{\tau_3 S+1} \qquad (6),$$

Then:

$$W_{3s} = \frac{ab(\tau_4 s+1)*[\tau_1\tau_4 S^2+(\tau_1+\tau_4)S+1+K_s a]*(\tau_2 s+1)*(\tau_3 s+1)}{[\tau_1\tau_4 S^2+(\tau_1+\tau_4)S+1+K_s a]S*(\tau_2 s+1)\{[\ldots]\}} ,$$

$$W_{3s}(S) = \frac{ab(\tau_4 s+1)(\tau_3 s+1)}{S\{[\ldots]\}} ,$$

$$W_{3s}(S) = \frac{M_c(S)}{X_n^*(S)} = \frac{ab(\tau_4 s+1)(\tau_3 s+1)}{S[D_4 S^4+D_3 S^3+D_2 S^2+D_1 S+1]} , \qquad (7)$$

$$M_c(S) = \frac{b M(S)}{\tau_2 s+1}; \qquad \{[\ldots]\}=D_4 S^4+D_3 S^3+D_2 S^2+D_1 S+1 \qquad (8)$$

Based on (7) and (8), we get:

$$M(S) = \frac{(\tau_2 s+1)}{b} * M_c = \frac{(\tau_2 s+1)}{b} * W_{3s}(S) * X_n * (S), \qquad (9)$$

$$[D_4 S^4+D_3 S^3+D_2 S^2+D_1 S+1]S * M(S) = a(\tau_3 S+1)(\tau_2 S+1)(\tau_4 S+1) * X_n.$$

The terms of sustainability of the enterprise are expressed by the next system of equations:

$$D_i > 0 ; D_3 D_2 > 2.25 D_1 D_4;$$

$$D_1 D_2 > 2.25 D_3 \qquad \qquad \left.\begin{array}{c}\\ \\ \\ \end{array}\right\} \qquad (10)$$

Next, the system is transformed into differential equations in the time domain.

The equation for the description of the dynamics of cost:

$$\ddddot{M}_c + d_3 \dddot{M}_c + d_2 \ddot{M}_c + d_1 \dot{M}_c + d_0 M_c = AX_n^* \qquad (11)$$

$$A = \frac{ab}{\tau_1 \tau_2 \tau_3 \tau_4} \quad , \quad \tau = 0.02 \quad . \qquad (12)$$

The equation for the description of revenue dynamics:

$$\dot{M} = -\frac{b}{\tau} M + \frac{\tau_2}{\tau} M_c + \frac{1}{\tau} \dot{M}_c \ . \qquad (13)$$

Enter the designations:

$$\left. \begin{aligned} M_c &= x_0 \\ \dot{M}_c &= x_1 = \dot{x}_0 \\ \ddot{M}_c &= x_2 = \dot{x}_1 \\ \dddot{M}_c &= x_3 = \dot{x}_2 \\ \ddddot{M}_c &= \dot{x}_3 \\ M &= x_4 \\ \dot{M} &= \dot{x}_4 \end{aligned} \right\} \qquad (14)$$

The system of differential equations in a normal form takes the form of:

$$\left. \begin{aligned} \dot{x}_0 &= x_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -d_3 x_3 - d_2 x_2 - d_1 x_1 - d_0 x_0 + -AX_n^* \\ \dot{x}_4 &= -\frac{b}{\tau} x_4 + \frac{\tau_2}{\tau} \dot{x}_1 + \frac{1}{\tau} x_0 \end{aligned} \right\} \qquad (15)$$

In this case, di are displayed on the basis of D_i (i=0...3).
For graphic interpretation, print: x_1, x_4, x_0 - x_4
$x_0 = M_c$, $x_4 = M$, $\Delta M = M_c - M$.

For simulation, parameters are set:

$$\begin{aligned} x_n^* &= 10^6 \\ a &= 1 \\ b &= 1.05 \\ \tau_1 = \tau_2 = \tau_3 = \tau_4 &= 1 \\ K_s &= 0.3 \\ K_{os} &= 0.4 \\ \tau &= 0.02 \end{aligned} \qquad (16)$$

Using this model and the Mathcad editor allowed us to obtain a graphical interpretation of transients. They are shown for the cost of production, revenue from sales of products and profits of the company, as well as the dynamics of indicators with open on K_s (parameters 16) and closed loops (parameters 16a) feedback in the control system SIF. Presented in figures 2, 3.
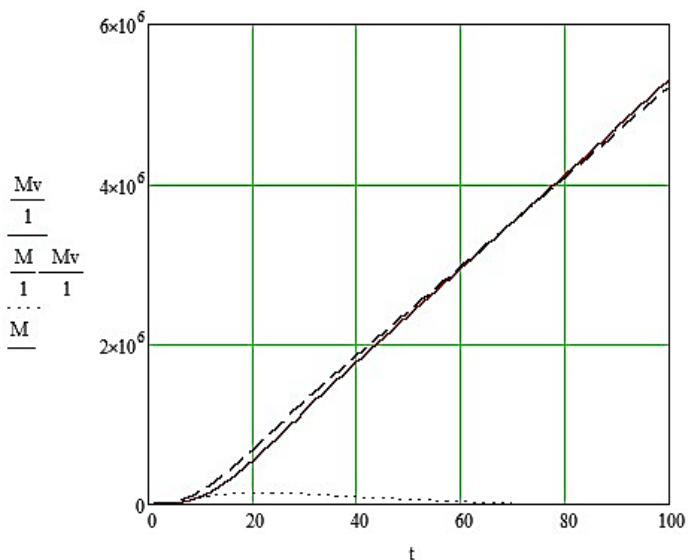
Fig. 2. Transients on the cost of production, revenue from sales of products and profits of the company.
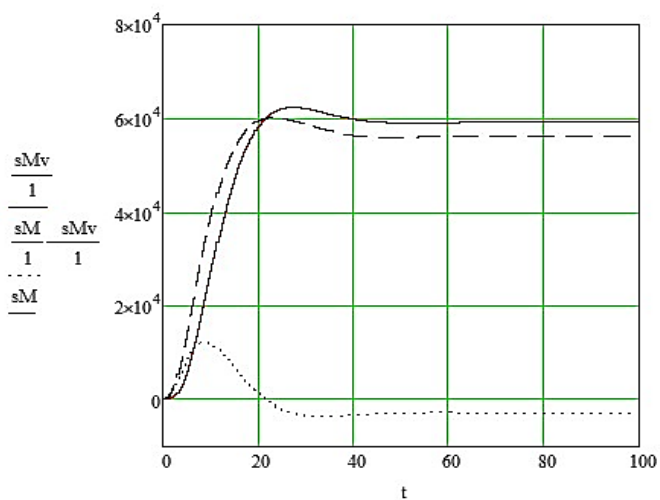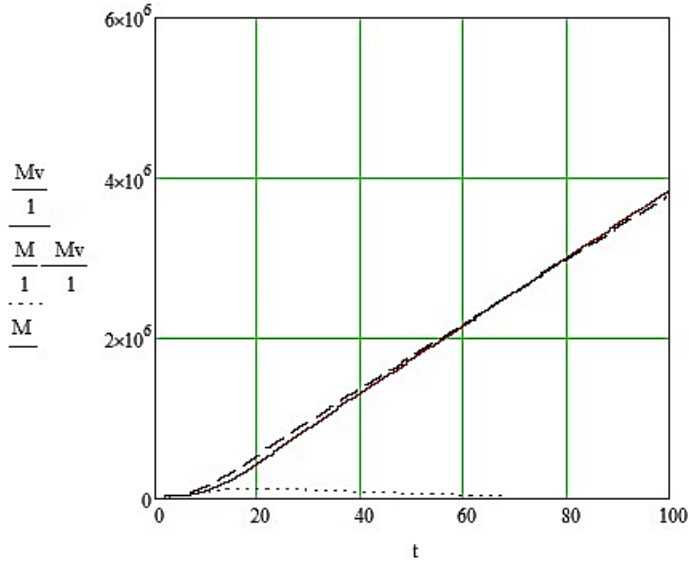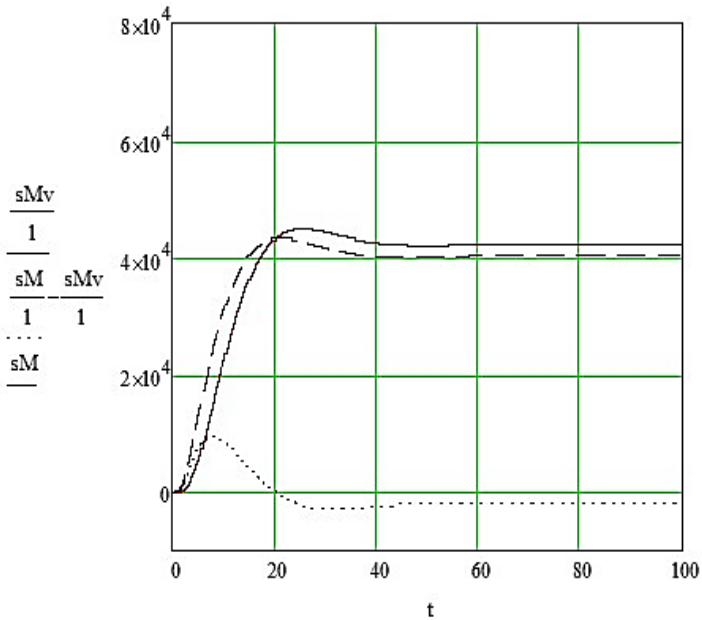

Fig. 3. Transients dynamics of economic indicators of the company.

$$K_c = 0.6 \quad K_{oc} = 0.5$$

(16 a)

a) Transients in the rate of change in the volume of revenue from sales and production costs.

b) Transients in the rate of change of profit of the company.

The graphs show the simulation results in the presence of an operating system (the control loop of the company is closed) and its absence (the control loop of the company, respectively, is open). The process is considered under conditions of perturbations determined by changes in the economic parameters of modeling.

In fig. 3. a) it is shown that with the open FL at cost, the rate of increase in cost is higher than with closed feedback, the deviation from the planned values increases, and the duration of the transition process, which leads to a decrease in management efficiency and an increase in the instability of the company's functioning. Fig. 3. b) shows a similar dependence of the rate of change of the firm's profit and the duration of the transition process on the state of the feedback loop in the SIF control system. Thus, mathematical modeling made it possible to clearly demonstrate the influence of the feedback state on the economic performance of the enterprise and transients.

4 Conclusions

The article provides a brief review and analysis of scientific papers on the issues and problems of mathematical modeling of activity and management of economic systems using the feedback principle. He made it possible to determine one of the directions in the development of modeling related to the interpretation of the nature of transients with a closed and open feedback loop in the company management system. Transients are determined by fluctuations in sales volumes and production costs in the presence of disturbing influences. The formulated initial ideas of the article made it possible to build a structural model of the functioning of a single-discipline firm, based on the scientific apparatus and ideology of TAC. The structural model clearly demonstrates the logic and content of the process of controlling the functioning of the SIF. Equations mathematically describing this process were derived. The study performed the transformation of the obtained expressions into a system of differential equations. This made it possible to build a mathematical model of transients. It is presented by revenue from sales and production costs, company profits, their totality with a closed and open loop operating system in the company management system. Using the Mathcad editor led to the desired graphical interpretation of the transients. This was a scientific task and determines the novelty of the article. Modeling clearly demonstrated the dependence of the dynamics of the main economic indicators of the company and the duration of the transition processes on the state of the feedback loop in the SIF control system. It reflected the mismatch of actual and planned indicators with an open FL. There is the positive role of feedback loops on the dynamics of the firm's functioning. At the same time, from a mathematical point of view, the direction of increasing the rates and profits of small businesses is the introduction of positive feedback loops in the range of values (0 ... - 0.5), as well as the combination of positive and negative feedback loops in terms of cost and revenue. The proposed mathematical model contributes to the further development of the methodology of mathematical modeling in the management of economic systems. It is aimed at increasing management efficiency by building a stable feedback loop. The circuit provides the necessary operational information for making and monitoring the implementation of various management decisions.

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