

Using the Mathematica package in teaching mathematical disciplines

Vladimir Grigoriev-Golubev¹, Natalia Vasileva^{2,*}, and Margarita Volodicheva³

¹Professor of Department of Mathematics, Saint-Petersburg State Marine Technical University, Saint-Petersburg, Russia

^{2,3}Associate Professor of Department of Mathematics, Saint-Petersburg State Marine Technical University, Saint-Petersburg, Russia

Abstract. This article analyzes the capabilities of the Wolfram Mathematica computer system, examines the feasibility of using its tools in the study of mathematical disciplines. The authors propose and demonstrate by examples a methodology for building a training course based on the integration of the methods of the discipline being studied and their implementation in the Mathematica environment. The paper explores the practical significance of including the Mathematica toolkit in the training course, which makes it possible to mathematically model various processes in modern society, demonstrate the solution of mathematically complex problems using the built-in functions of the package, as well as provide visualization of analytically obtained solutions.

1 Introduction

In the professional activity of a specialist of any engineering profile, the ability to solve complex and mathematically time-consuming tasks is required. Competitiveness in the labor market of university graduates in modern society is determined not only by the level of their basic training, but also by the ability to use computer technology in their professional activities. Therefore, in order to improve the quality of engineering education and the effectiveness of the process of mathematical training of specialists, it is advisable to include in the educational process work with universal software products, in particular with currently widely used applied mathematical packages: Mathcad, MATLAB, Mathematica. The skills of using computer technology in the educational process should be formed in students from the first year of study. The most effective in this case is the construction of a system of mathematical training of specialists based on the integration of mathematics and computer science, which is becoming increasingly relevant with the development of informatization of society. Currently, a new discipline has appeared, created at the junction of mathematics and new information technologies – computer mathematics, without the tools of which it is impossible to build a modern educational process that includes three components: educational, scientific and practical.

* Corresponding author: vasileva_n2017@mail.ru

The presence of three groups of tools in modern mathematical application software packages: symbolic mathematics, computational mathematics and graphics, makes it possible to easily carry out analytical transformations of mathematical expressions, quickly perform complex calculations and present the results graphically. All this simplifies routine computing work, makes it more attractive [1].

The use of mathematical packages creates not only a convenient computing, but also a flexible educational environment of the discipline being studied, where educational technologies mean not only the use of elements of computer mathematics, but also the use of all means of modern information technologies, including a set of methods for receiving and processing information using modern means of communication, computer technology and software [2].

If a few years ago Mathcad, MATLAB and Maple packages were the most used in the educational process [3,4], now the world leader in the use of universal computer mathematics systems is the Mathematica package, which has the highest computing speed and a huge set of useful functions [5-7].

2 Using the Mathematica package in teaching mathematical disciplines

The basis of computer mathematics is the combination of the presentation of the methods of the discipline being studied and the demonstration of their implementation in some universal system of computer mathematics, in this case in the Mathematica package. A large number of built-in computing tools in the MATHEMATICA system allows you to solve many problems in a language that is extremely close to the usual language of mathematical calculations. The latest versions (11,12) have the function of automatic prompts, which suggests further actions to the user depending on the context of the work and allows you to select the most appropriate operations from those that the package supports.

The need to build a training course using computer mathematics technologies appears where the solution of problems is associated with large cumbersome calculations or where analytical solutions to problems are obtained in the form of superpositions of special functions or in the form of series, since in these cases it is sometimes impossible to investigate the resulting solution, to understand its physical meaning. There are three sections of the higher mathematics course in which the use of computer mathematics technologies is urgently needed: computational mathematics, mathematical physics and discrete mathematics.

2.1 Numerical methods in the Mathematica environment

Each section of the course "Computational Mathematics" requires the use of software tools. The Mathematica system includes a huge number of built-in functions, sufficient to provide software support for this course. The Mathematica system also has extensive graphical capabilities from the construction of two-dimensional and three-dimensional graphs to dynamically changing curves and surfaces. For example, with an approximate solution of the system

$$\begin{cases} x^2 + y^2 - 3 = 0, \\ 3x + \sin(x - y) = 0 \end{cases} \quad (1)$$

it is required to set the initial approximation and specify the range on which to find its solutions. In this case, a graphical solution of the system should be carried out beforehand (Fig.1).

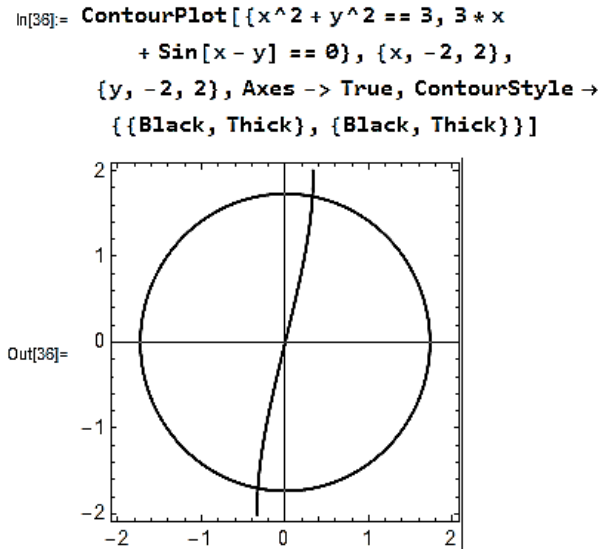


Fig. 1. Graphical solution of the system.

Based on the graphical (rough) solution, you need to set the interval in which both solutions of the system are contained. They are defined by the FindRoot function together with the Table function (Fig.2). The program specifies both solutions, as well as the singularity point (0,0) that falls within the specified interval.

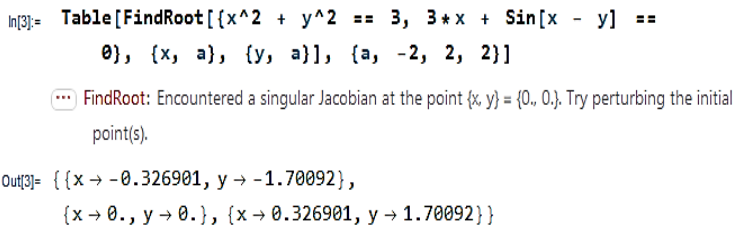


Fig. 2. All solutions of the system.

Using the Mathematica package is especially effective when studying the sections of the training course related to large cumbersome calculations. For example, in the section "Interpolating functions", after outlining the various forms of the interpolation polynomial and x methods of approximation of a tabulated function, it is enough to give an example of constructing an interpolation polynomial in the Mathematica package, since in this package the polynomial approximation is carried out using the Interpolating Polynomial function, which itself selects the interpolation polynomial depending on the properties of the tabulated function. Moreover, the interpolation polynomial is displayed on the screen in standard form (Fig. 3), which is impossible in other systems of computer mathematics.

```

data := {{0.25, 1.85}, {0.5, 2.85}, {0.75, 3.9}, {1, 5},
         {1.25, 6.15}, {1.5, 7.34}}

p[x_] = Collect[InterpolatingPolynomial[data, x], x]

0.91 + 3.60867 x + 0.7 x2 - 0.453333 x3 + 0.32 x4 - 0.0853333 x5
    
```

Fig. 3. The interpolation polynomial in the standard form.

At the same time, the training material comments on the package syntax and options in referring to the built-in function used in the above listing, for example

$$\text{Interpolating Polynomial [data, x],} \tag{2}$$

where data is the list by which the polynomial is constructed, x is the variable of the polynomial. The Collect function [P, x] reduces the interpolation polynomial to the standard form.

The presence in the Mathematica system of the built-in functions JordanDecomposition – obtaining the Jordan form of the matrix and the Jordan basis and MatrixExp - calculating the exponent from the matrix, allows you to find an exact solution to a linear system of large-order differential equations (Fig. 4). The Jordan form of the matrix is also used in the calculation of functions from matrices and the study of graphs for isomorphism. Recently, there have been works in which Jordan forms of matrices are used in the processing of graphic signals to study the structural and functional properties of various networks, which makes it possible to significantly simplify the processing of their numerical data. This is shown in [8] by the example of calculating the numerical characteristics of the highway network in New York.

```

Remove[A, c1, s, q, pc, p1, p01, p00]
A = {{-1.1, 0.1, 0.1, 0.2, 0.1, 0.2}, {0.2, -0.1, 0, 0, 0, 0}, {0.1, 0, -0.1, 0, 0, 0},
     {0.4, 0, 0, -0.2, 0, 0}, {0.1, 0, 0, 0, -0.1, 0}, {0.3, 0, 0, 0, 0, -0.2}};
{s, Aj} = JordanDecomposition[A];
Aj // MatrixForm
q[t] = MatrixExp[Aj t];
pc[t] = s.q[t] // Simplify;
p01 = {1, 0, 0, 0, 0, 0};
p00 = Evaluate[pc[t]] /. t -> 0;
c1 = LinearSolve[p00, p01]

Out[22]= {-0.975444, -7.85046 × 10-17, 0.155495, -1.44461 × 10-16, -7.14801 × 10-17, -0.428242}
    
```

Fig. 4. Determination of the exact solution of a linear system of differential equations.

2.2 Solving problems of mathematical physics in the Mathematica system

The problems of mathematical physics, which, as a rule, lead to the solution of partial differential equations, are quite complex problems. The equations of mathematical physics model a variety of physical processes in hydrodynamics, elasticity theory, aerodynamics and other fields. The solution obtained by analytical methods is most often presented in the form of an integral that is not taken in elementary functions, or in the form of a series, or expressed in terms of special functions. This kind of solution does not make it possible to understand how it describes a physical process and to investigate it depending on time. Therefore, when

studying a course of mathematical physics related to mathematical modeling tasks, it is especially important to use a computer mathematics system, implementing the learning process according to the scheme: description of the physical process – mathematical model – computational algorithm – program [9].

In the problem where it is required to find the temperature distribution of a rod of length l , the ends of which are heat insulated, and at the initial moment of time the heat distribution is described by a function $f(x) = x$, the physical process is described by the heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{3}$$

with boundary and initial conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0, \quad u(x,0) = x. \tag{4}$$

By the Fourier method, the analytical solution is obtained in the form of a series

$$u(x,t) = \frac{l}{2} - \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{\pi(2n-1)x}{l} e^{-\frac{\pi^2(2n-1)^2 a^2 t}{l^2}}, \tag{5}$$

and visualization of the solution can be obtained using the tools of the Mathematica system (Fig. 5). To do this, a function is described, given by formula (5), in which the series is limited to ten terms. The graph of this function is published using the built-in Plot function, designed for plotting. The options of this function control the color, the thickness of the curve, set the range of presentation of the results, and also make it possible to note which parameter values correspond to each curve.

From the constructed graphs of the solution (Fig. 5) at time points $t = 0$, $t = 0,5$, $t = 1$ the physical process described in the problem is clear. If at the initial moment of time the temperature of the rod changes according to a linear law, then already at $t = 1$ it is distributed evenly, and at large times $t \geq 10$ it becomes the same, equal to 1.

```

In[ ]:= Remove[u, t, x]
u[x_, t_] =
1 - 8 / π^2 * Sum[1 / (2 n - 1)^2 * Exp[- (2 n - 1)^2 * π^2 * t / 4] * Cos[ (2 n - 1) * π * x / 2];
Plot[{u[x, 0], u[x, 0.5], u[x, 1], u[x, 10]}, {x, 0, 2}, AxesLabel -> {"x", "u"},
PlotStyle -> {{Black, Thick}, {Black, Thick}, {Black, Thick}}, PlotRange -> {0, 2},
Epilog -> {Text[TraditionalForm["t=0"], {1.8, 1.9}],
Text[TraditionalForm["t=0.5"], {1.8, 1.35}],
Text[TraditionalForm["t=1"], {1.8, 1.1}],
Text[TraditionalForm["t=10"], {1.8, 0.9}]}]
    
```

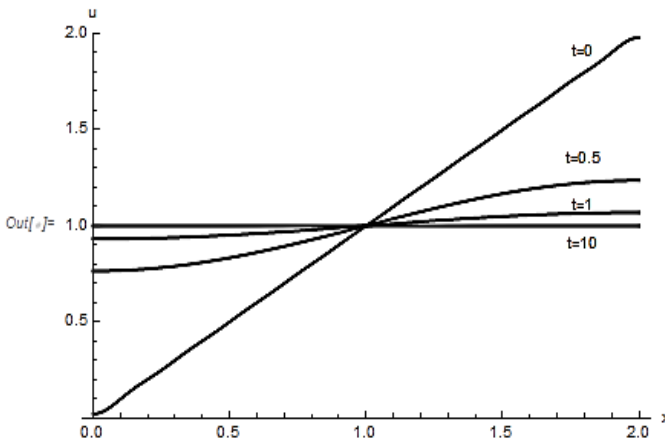


Fig. 5. The distribution of the rod temperature at $t = 0, t = 0,5, t = 1, t = 10$.

It is very useful to show different ways of solving the same problem. For example, the latter problem can be solved directly with the tools of the Mathematica package, using the built-in DSolve or DSolvevalue functions (Fig. 6).

```

In[45]:= Remove[u, t, x, sol, bc, ic, app, heqn]
l = 2;
a = 1;
heqn = D[u[x, t], t] == a^2 * D[u[x, t], {x, 2}];
ic = u[x, 0] == x;
bc = {Derivative[1, 0][u][0, t] == 0, Derivative[1, 0][u][l, t] == 0};
sol = DSolve[{heqn, ic, bc}, u[x, t], {x, t}] /. {K[1] -> m}
    
```

$$\text{Out[51]} = \left\{ \left\{ u[x, t] \rightarrow 1 + \sum_{m=1}^{\infty} \frac{4 (-1 + (-1)^m) e^{-\frac{1}{4} m^2 \pi^2 t} \text{Cos}\left[\frac{m \pi x}{2}\right]}{m^2 \pi^2} \right\} \right\}$$

Fig. 6. Solving the problem of the distribution of the rod temperature using the package function.

Many physical processes are described by the wave equation of mathematical physics. The wave equation is found in problems of transverse vibrations of strings, longitudinal vibrations of rods, electrical vibrations in wires.

For example, the vibrations of an infinite string, the initial profile of which is given by the function

$$\varphi(x) = \begin{cases} 1 - x^2, & x \in [-1, 1], \\ 0, & x \notin [-1, 1] \end{cases} \quad (6)$$

and there are no initial velocities, described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (7)$$

with initial conditions

$$u(x, 0) = \varphi(x), \quad \frac{\partial u}{\partial t}(x, 0) = \psi(x) = 0. \quad (8)$$

The analytical solution of the problem (7,8) is obtained in the form of the D'Alembert formula

$$u(x, t) = \frac{1}{2}(\varphi(x + at) + \varphi(x - at)) + \frac{1}{2a} \int_{x-at}^{x+at} \psi(s) ds, \quad (9)$$

which, under the given initial conditions (8), takes the form

$$u(x, t) = \frac{1}{2}(\varphi(x + at) + \varphi(x - at)). \quad (9)$$

Figure 7 shows the solution described by formula (9) and the assignment of functions $\varphi(x)$ by formula (6) and $\psi(x)$ from the initial conditions (8). This solution describes the profile of the string at a time t . The view of the string profiles at time points from 0 to 2 with a step of 0.4 is demonstrated in the form of a gallery of drawings, which is obtained using the Table function together with the Plot and Evaluate functions.

Such a gallery of drawings clearly shows the physics of the process. The above figures show how the transformation of a wave occurs in an infinite string: one wave, which at the initial moment of time is a superposition of two waves, then splits into two waves, then they move in different directions.

The process of changing the shape of the string is even better and more clearly observed if you use the Animate function, which is described in detail in [7]. With this function, you can build an animated graph, demonstrating a consistent change in the string profile. Unfortunately, it is impossible to show the animated construction of curves and surfaces in a paper version. This can be demonstrated only in the remote form of lecturing.

```
In[82]:= Remove[u, t, x, φ, ψ]
φ[x_] = If[Abs[x] < 1, 1 - x^2, 0];
ψ[x_] = 0;
u[x_, t_] = (φ[x - t] + φ[x + t]) / 2;
Table[Plot[Evaluate[u[x, t]], {x, -5, 5}], PlotRange -> {0, 1.2}, PlotStyle -> {Black, Thick}],
{t, 0, 2, 0.4}]
```

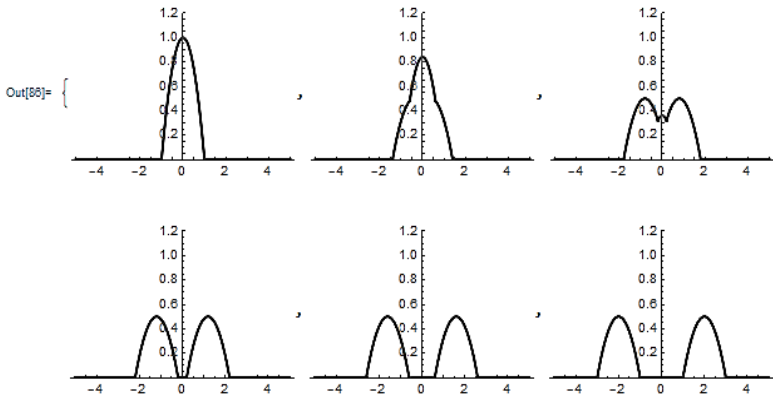


Fig. 7. Gallery of infinite string profiles at $0 \leq t \leq 2$.

In some problems of mathematical physics, obtaining an analytical solution is associated with great, sometimes insurmountable difficulties, due to the need to solve transcendental equations or calculate integrals that are not taken in elementary functions. In these cases, it is possible to carry out the solution by the method of separation of variables, using directly the tools of the Mathematica package. Many similar examples are described and their solutions are shown in [7].

2.3 Using the Mathematica package in the study of discrete mathematics

Discrete mathematics studies discrete mathematical structures and includes such sections as combinatorics, mathematical logic, graph and network theory, coding theory, etc., which are widely used in the field of computer technology and automatic control.

The Mathematica computer system has great capabilities in solving problems of logic algebra. With its help, you can find a vector of values for a function given by a formula, represent the function in perfect disjunctive normal form (PDNF), perfect conjunctive normal form (PCNF), get a superposition of functions, exclude fictitious variables, construct a dual function with respect to a given function, represent the function as a Zhegalkin polynomial, write the function in various bases, in particular in the basis of Schaeffer functions and in the basis of Webb functions, etc.

The most important task of logic algebra is to minimize the Boolean function, which allows you to simplify the schemes. Among all the packages of mathematical programs, only Mathematica successfully solves this problem. For example, when minimizing a function given by a vector of values

$$f(x, y, z, u) = (0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1),$$

PDNF which has the form

$$f(x, y, z, u) = \bar{x} \cdot \bar{y} \cdot \bar{z} \cdot \bar{u} \vee \bar{x} \cdot \bar{y} \cdot z \cdot \bar{u} \vee \bar{x} \cdot y \cdot \bar{z} \cdot \bar{u} \vee \bar{x} \cdot y \cdot z \cdot \bar{u} \vee x \cdot \bar{y} \cdot \bar{z} \cdot \bar{u} \vee x \cdot \bar{y} \cdot z \cdot \bar{u} \vee x \cdot y \cdot \bar{z} \cdot \bar{u} \vee x \cdot y \cdot z \cdot \bar{u}$$

it turns out a simpler function

$$g(x, y, z) = x \cdot y \vee \bar{y} \cdot z.$$

In the Mathematica environment, you can build images of graphs of various types: digraphs, non-digraphs, multigraphs, pseudographs. Using Mathematica, you can find a union, intersection, ring sum, direct product of graphs; construct an adjacency matrix, an incidence matrix, a Kirchhoff matrix for a graph, restore a graph from a given adjacency matrix; you can find metric characteristics of a graph, blocks and bridges in a graph, the number of spans in a graph, connected components of a graph, find the number of edge connectivity, the chromatic number of the graph, conduct a graph study on the existence of Eulerian and Hamiltonian cycles in the graph.

The construction of Hamiltonian cycles is an integral part of solving the problem of a traveling salesman who must visit a finite number of settlements along the shortest route and then return to the starting point. Using the Mathematica package, it is possible to find the shortest route connecting two given points, which is used in solving a large number of practical problems, in particular when finding the shortest path using on-board computers of modern cars and when determining the route of message delivery from one server to another using a router of the Internet computer network. Finding optimal routes using Mathematica allows you to determine the scheme of building roads connecting settlements, for which the cost of construction will be minimal.

Using the Mathematica computer system, it is possible to examine graphs for isomorphism, to find all graphs that are isomorphic to a given one, which is described in detail in [10]. Such a study allows us to establish the structural similarity or structural difference of objects that are modeled by graphs. With the help of Mathematica, it is possible to study graphs for planarity, which is important for radio electronics, since in the manufacture of printed circuits, individual conductors are not isolated and therefore should not intersect.

3 Discussion of the results of the study

The use of the Mathematica computer system as a basic software product of the educational process makes it more modern, corresponding to the level of development of science and meeting the requirements of practice. On the one hand, some sections of the course can be used, abandoning the methods that are routine at this stage of computer technology development. For example, it is possible not to consider the Hurwitz criterion and the construction of the Mikhailov curve in stability theory in the training course, since the Mathematica package calculates the roots of a polynomial of any degree. On the other hand, the extensive capabilities of the Mathematica system make it possible to include practical tasks in the course being studied, not limited to tasks with simple calculations.

The integration of mathematics and computer science into computer mathematics involves not just teaching methods of mathematics and computer technologies within the same department of Mathematics, this means combining the tools of both disciplines in one course, starting with lecture material. Such a format of lectures makes it possible to bring the solution of each demonstrated problem to a number, allows for a good visualization of solutions, with which students can understand the described physical process.

With this form of training, the teacher, demonstrating program listings at lectures, introduces students to the syntax of the package, explains the meaning of all the options of the built-in functions used. Such a gradual, step-by-step study turned out to be very effective, since new information is associated with solving a certain range of tasks, which contributes to better understanding and memorization. Such a construction of the educational process

justifies itself even when the course includes a set of practical (laboratory) classes. Students, coming to a practical lesson, are already familiar not only with the method used, but also with the means that need to be used. In this case, tutoring is enough to conduct classes in the computer room.

The training course in the format of "computer mathematics" assumes that the lecturer not only introduces students to the technologies of a computer system, but also draws attention to its features that need to be taken into account in specific calculations and information about which cannot be found in any literature. For example, as the limit of a function at a given point (which does not always exist) Mathematica gives a right-hand limit by default. The results of solving some linear algebra problems using Mathematica (finding eigenvalues and eigenvectors, constructing a Jordan form) for matrices with a large number of conditionalities and large skew coefficients depend on the input data input format. Moreover, it is necessary to acquaint students with how to overcome the problems that arise in these cases. Of course, the teacher must master the computer system himself, which is rather a positive factor than a negative one, since the acquired amount of knowledge can be successfully used in scientific work.

4 Conclusion

We live in a dynamically developing world. In practice, new processes appear that require study, and hence the construction of a mathematical model. In this regard, mathematical sciences such as linear and abstract algebra, Boolean algebra, the theory of random processes, mathematical physics, and many others are developing by leaps and bounds. This should be reflected in education, primarily in mathematics, strengthening its scientific and practical components, which, in turn, requires course software, and one of the most powerful software tools currently is the continuously developing Mathematica computer system.

References

1. N.V. Vasileva, M.I. Volodicheva, S.N. Leora, Scientific notes of the ISGZ **12(1)**, 150 (2014)
2. R. Luppardini, J. ET& S **8.3**, 103 (2005)
3. N.V. Vasileva, V.V. Grigoriev-Golubev, I.V. Evgrafova, J. Marine Intellectual Technologies **1.2**, 75 (2017)
4. N.V. Vasileva, V.V. Grigoriev-Golubev, I.V. Evgrafova, N.B. Kunturova, S.N. Leora, J. Marine Intellectual Technologies **3.4**, 158 (2017)
5. M.I. Volodicheva, V.V. Grigoriev-Golubev, M.A. Kutejnikov, J. of Marine Intellectual Technologies, Saint-Petersburg **1.4**, 158 (2019)
6. V.R. Kristalinskij, R.E. Kristalinskij, J. Modern information technologies and IT education **15.4**, 981 (2019)
7. N.V. Vasileva, M.I. Volodicheva, V.V. Grigoriev-Golubev, *Computer mathematics in computational problems*, (SPbGMTU, Saint-Petersburg, 2020)
8. J.A. Deri, J.M.F. Moura, IEEE J. Selected Topics in Signal Processing **11.6**, 785 (2017).
9. A.A. Samarsky, A.P. Mikhailov, *Matematicheskoe modelirovanie. Idei. Metody`. Primery`*, (Fizmatlit, Moscow, 2001)
10. M.I. Volodicheva, S.N. Leora, J. Applied discrete mathematics **40**, 87 (2018)