

On the training of scientific personnel at post-graduate school

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Abstract. The possibilities of modern packages of symbolic mathematics for the scientific work of novice researchers in mathematics are discussed. Counterexamples for Schwartz theorem on the equality of mixed derivatives are considered. It is shown how you can use the graphical and computational capabilities of symbolic mathematics packages for their analysis so that novice researchers with the traditional engineering background in mathematics understand them. The article focuses on the problems of ambiguous perception of the PC user of three-dimensional graphic images on the monitor screen. This is important for operators who make decisions based on information obtained in a mixed virtuality or mixed reality environment. The analysis of counterexamples for the Schwartz theorem is carried out using both classical mathematics and possibilities of symbolic mathematics packages. It is shown that this assertion is not necessary. During the virtual demonstration of three-dimensional graphic objects, recommendations on their perception in terms of information environments are provided.

1 Introduction

The training of scientific personnel involves the course of higher mathematics which, among other things, develops logical culture and mathematical thinking. However, the level of mathematics knowledge of novice researchers is often low. Moreover, many of them are convinced that good knowledge of mathematics can be replaced by digital technologies. No one can deny achievements of science and practice obtained with the help of modern digital technologies. These innovations make it possible to replace analytical proofs and supplement mathematical concepts, definitions and methods with graphic images. Working with graduate engineering students, we offer tasks in mathematics that involve a deep mathematical analysis and the use of symbolic mathematics packages (SMP) to show possibilities and need for mathematical methods and digital technology tools. These tasks are given to students and cadets which makes it possible to select the most capable ones for research work. The current study will show how to master complex materials with the help of visualization, using SMPs and their abbreviated versions. An attempt was made to show the subtleties of the Schwartz theorem [1] on the equality of second-order mixed derivatives for functions of two variables. This work is a continuation of the studies described in [2]. There are no such examples in textbooks for non-mathematical specialties, because it is considered that this is the subject of university courses in mathematics, where these examples are provided without comments.

This approach causes confusion of students about the equality of mixed derivatives. It will be shown how the graphical and computational capabilities of the SMP can be used to analyze conditions of the Schwartz theorem. In addition, we will pay attention to the problems of ambiguous perception of three-dimensional graphic images on the monitor screen by a PC user [2-5]. This is especially important for operators who make decisions based on virtual information "diluted" with elements of reality. The problem of insufficient mathematical training and skills in the use of digital technologies is of interest to scientific supervisors, postgraduate departments, and the administration of universities.

2 Materials and Methods

The analysis of counterexamples related to the Schwartz theorem helps to solve the problem. In mathematics, the equality of mixed derivatives is the ability to change the order of differentiation when calculating mixed partial derivatives of the same order without losing their equality. The mixed second order derivatives satisfy the equality under certain conditions.

$$\frac{d^2f}{dxdy} = \frac{d^2f}{dydx} \quad (1)$$

The conditions for the fulfillment of equalities of type (1) were dealt with by Schwartz, Clairaut, Jung and other well-known mathematicians for more than 150 years. The story full of drama can be found in the literature. In 1881 Schwartz succeeded in giving the first rigorous proof of equality (1) [1]. For analytic functions,

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the equality of mixed derivatives easily follows from their representation by power series. Consider the general case.

Schwartz theorem. Let the following conditions be met:

1. $z = f(x, y)$, $\frac{df}{dx}$, $\frac{df}{dy}$, $\frac{d^2f}{dxdy}$, $\frac{d^2f}{dydx}$ are determined in some neighborhood of the point (x_0, y_0) .
2. $\frac{d^2f}{dxdy}$, $\frac{d^2f}{dydx}$ are continuous at point (x_0, y_0) .

Then $\frac{d^2f(x_0, y_0)}{dxdy} = \frac{d^2f(x_0, y_0)}{dydx}$, that is, the mixed second order derivatives are equal at every point where they are continuous.

The theorem on the equality of the second mixed partial derivatives extends inductively to mixed partial derivatives of higher orders, provided that they are continuous.

The paper discusses the representation of three-dimensional mathematical objects (surfaces of example functions for the Schwartz theorem) on the plane (PC screen) using graphic programs and new concepts: virtuality, augmented reality, mixed reality [4-5]. These concepts are necessary for the selection and construction of a flat image, which makes it possible to obtain an unambiguous perception of drawings as three-dimensional objects. This approach to the presentation of information and its image is being developed wherever information comes in a virtual form on computer screens.

The search for examples helped the authors come up with their own examples. When analyzing the examples found in the literature and on websites, we added calculations, corrected inaccuracies and typos, found intervals of limiting values when studying discontinuities of second derivatives. The SMP-based graphics and calculations are used. Only one example is given and studied in the provided. In contrast to the example provided in [2], it is of general nature. With the help of two parameters included in the task, we can get both the same second mixed derivatives and those that do not match. Moreover, in both cases the mixed derivatives are not continuous. The first proves that Schwarz theorem is not necessary. Consider the function

$$f(x, y) = \frac{|ax+by|^3}{\sqrt{x^2+y^2}}, \quad x^2 + y^2 > 0, \quad (2)$$

where a, b denote simultaneously non-zero parameters. Surface plots of this function for a=2 and b=1

$$f(x, y) = \frac{|2x+y|^3}{\sqrt{x^2+y^2}} \quad (3)$$

Can be seen in Figures 1-2. Formula (3) determines the continuous function everywhere on the plane except for the origin. Prove that at point (0,0) it can be extended by a zero value, preserving continuity. On all surface graphs of function (3), the program determines the function by the value at point (0,0). Making a polar substitution, we have:

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{|ax+by|^3}{\sqrt{x^2+y^2}} = \lim_{r \rightarrow 0} r^2 |\text{acos}\varphi + \text{bsin}\varphi|^3 = 0 \quad (4)$$

Write down the continuous function for any x,y without introducing a new function notation, i.e.:

$$f(x, y) = \begin{cases} \frac{|ax+by|^3}{\sqrt{x^2+y^2}}, & x^2 + y^2 > 0; \\ f(0,0) = 0. & \end{cases} \quad (5)$$

In Fig. 1, the surface is unambiguously perceived in the form of a “gutter”. However, for Fig. 2 this is not true. To the same observer, the same surface in Fig. 2 appears to be an inverted or vertical “gutter”. Different perceptions of graphic (virtual) information are not the cause of trouble. This is perceived as some kind of illusion. However, in decision-making systems such collisions can lead to accidents and even catastrophes [3-4]. Therefore, in Fig.1 the program offers a grid, a chiaroscuro and a convenient angle for observation, and in Fig. 2 it offers a grid, a chiaroscuro and a fill, but not a good angle for the observer. In terms of informational perception, the virtual representation of the spatial object requires certain elements for the correct perception (recognition or full vision) of a virtual object on the plane (two-dimensional virtual space), when this object is in a “real” three-dimensional space, albeit in ideal mathematical terms. The addition of the elements is called augmented reality, and together with the virtuality on the PC screen, this information environment (screen, plane) is called mixed reality. Of course, mathematical graphical models cannot be considered real, but the benefits of using the theory of information environments and augmented reality in mathematical education and science are undeniable. We do not consider numerous cases of accidents and disasters before the advent of electronic means of transmitting visual information, the reasons for which were the result of an unequal and ambiguous perception of the visual information of the environment by a person without technical “intermediaries”.

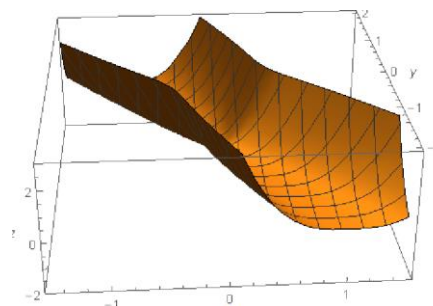


Fig. 1. The surface of the function under study, which is uniquely defined by the observer as a “trough”

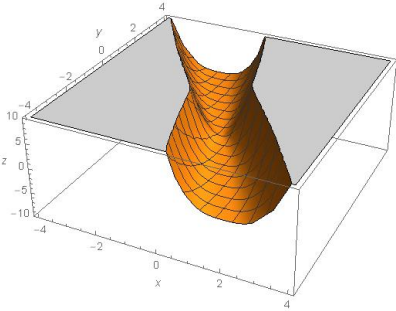


Fig. 2. The same surface (Fig. 1) is not uniquely determined by the observer

Here (Fig.1–2) and further (Fig. 3–7) all surfaces are built for $a = 2$ and $b=1$.

Calculate the first derivatives of function (2) outside the origin. The “module cube” function is twice differentiable, and its first and second derivatives are calculated by formulas:

$$\frac{d}{dx} |x|^3 = 3x|x| \quad (6)$$

$$\frac{d^2}{dx^2} |x|^3 = \frac{d}{dx} (3x|x|) = 6|x| \quad (7)$$

The first derivative with respect to x of function (2) has the form

$$f'_x(x, y) = 3a \frac{(ax+by)|ax+by|}{\sqrt{x^2+y^2}} - \frac{x}{(\sqrt{x^2+y^2})^3} |ax+by|^3. \quad (8)$$

The surface of the derivative with respect to x of function (2) for $a=2$ and $b=1$

$$f'_x(x, y) = 6 \frac{(2x+y)|2x+y|}{\sqrt{x^2+y^2}} - \frac{x|2x+y|^3}{(\sqrt{x^2+y^2})^3}$$

Can be seen in Fig. 3.

One can discuss the choice of an angle and the preparation of graphs of first derivatives using augmented reality elements for the unambiguous and correct perception of the surface on the PC screen by different observers as a spatial model

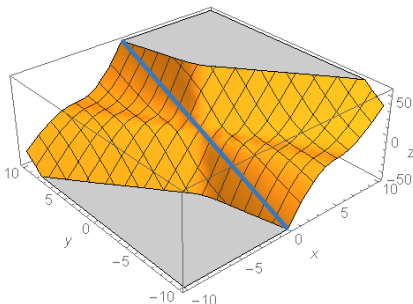


Fig. 3. The surface of the first derivative of the function along the x axis

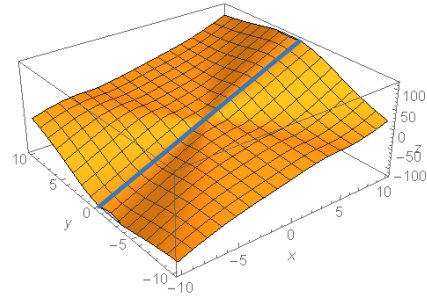


Fig. 4. The surface of the first derivative of the function along the x axis along the y axis

Similarly, the derivative with respect to the same function has the form

$$f'_y(x, y) = 3b \frac{(ax+by)|ax+by|}{\sqrt{x^2+y^2}} - \frac{y}{(\sqrt{x^2+y^2})^3} |ax+by|^3 \quad (9)$$

The surface of the derivative with respect to function (2) for $a=2$ and $b=1$, i.e.

$$f'_y(x, y) = 3 \frac{(2x+y)|2x+y|}{\sqrt{x^2+y^2}} - \frac{y|2x+y|^3}{(\sqrt{x^2+y^2})^3}$$

can be seen in Fig. 4.

On all graphs of the surfaces of the first derivatives (Fig. 3, 4), the program determines the first derivatives at point $(0,0)$. Calculate the first derivatives for function (5) at the origin:

$$f'_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \rightarrow 0} \frac{|ax|^3}{x|x|} = 0. \quad (10)$$

Similarly, $f'_y(0,0) = 0$, i.e. both first derivatives can be extended by zero value, because the double limits of these derivatives at point $(0,0)$ are also equal to zero. In fact,

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f'_x(x, y) = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \left(6 \frac{(2x+y)|2x+y|}{\sqrt{x^2+y^2}} - \frac{x|2x+y|^3}{(\sqrt{x^2+y^2})^3} \right) =$$

$$\lim_{r \rightarrow 0} r((6(2 \cos(\varphi) + \sin(\varphi))|2 \cos(\varphi) + \sin(\varphi)|) - \sin(\varphi)|2 \cos(\varphi) + \sin(\varphi)|^3) = 0. \quad (11)$$

Similarly, for another first derivative:

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f'_y(x, y) = \lim_{r \rightarrow 0} r(3(2 \cos(\varphi) + \sin(\varphi))|2 \cos(\varphi) + \sin(\varphi)| - \cos(\varphi)|2 \cos(\varphi) + \sin(\varphi)|^3) = 0 \quad (12)$$

Formulas (13), (14) give an analytical representation of the continuous first partial derivatives of the function at all points of the coordinate plane for non-zero values a, b :

$$\begin{cases} f'_x(x, y) = 3a \frac{(ax+by)|ax+by|}{\sqrt{x^2+y^2}} - \frac{x}{(\sqrt{x^2+y^2})^3} |ax+by|^3, \\ x^2 + y^2 > 0; \\ f'_x(0,0) = 0. \end{cases} \quad (13)$$

$$\begin{cases} f'_y(x, y) = 3b \frac{(ax+by)|ax+by|}{\sqrt{x^2+y^2}} - \frac{y}{(\sqrt{x^2+y^2})^3} |ax+by|^3, \\ x^2 + y^2 > 0; \\ f'_y(0,0) = 0. \end{cases} \quad (14)$$

All the visualizations of surfaces are built with $a = 2$, $b = 1$.

When calculating derivatives, routine analytical work is often performed using symbolic mathematics packages, which leads to the loss of “manual” calculation skills. Therefore, the use of mathematical software packages is wrong and even harmful. For example, performing manual calculations of second-order mixed derivatives, we will not see $|ax + by|$ in the denominator as in (15, 16). Having the skills of analytical transformations, it is not difficult to figure out what is going on. Surprisingly, these derivatives in (15, 16) do not lose continuity at points of the line $ax + by = 0$ outside the origin, if we extend their definition at these points. Using formulas (6-9), we can obtain the same mixed derivatives outside the origin in general form

$$f''_{xy}(x, y) = f''_{yx}(x, y) = \frac{3(xy(ax+by)^4 - (a^2+b^2)xy(ax+by)^2(x^2+y^2) + ab(ax+by)^2(x^2+y^2)^2)}{(x^2+y^2)^{\frac{5}{2}}|ax+by|} \quad (15)$$

It is obvious that the second derivatives (15) coincide outside the point (0,0). This follows from the Schwartz theorem, because the derivatives are continuous.

Calculate the mixed derivatives at the origin using formulas (13.14):

$$\begin{aligned} f''_{xy}(0,0) &= \lim_{y \rightarrow 0} \frac{f'_x(0, y) - f'_x(0,0)}{y} = \\ &= \lim_{y \rightarrow 0} \frac{1}{y} \left(\frac{3aby|by|}{|y|} \right) = 3ab|b|; \end{aligned} \quad (16)$$

$$\begin{aligned} f''_{yx}(0,0) &= \lim_{x \rightarrow 0} \frac{f'_y(x,0) - f'_y(0,0)}{x} = \\ &= \lim_{x \rightarrow 0} \frac{1}{x} \left(\frac{3abx|ax|}{|x|} \right) = 3ab|a|. \end{aligned} \quad (17)$$

It is easy to see that at point (0,0) the mixed derivatives are not equal if:

$$|a| > 0, |b| > 0, |a| \neq |b|. \quad (18)$$

The mixed derivatives outside the origin with respect to x and y from function (5) for $a=2$ and $b=1$ are calculated by formula (19)

$$f''_{xy}(x, y) = \frac{3(xy(2x+y)^4 - 5xy(2x+y)^2(x^2+y^2) + 2(2x+y)^2(x^2+y^2)^2)}{(\sqrt{x^2+y^2})^5 |2x+y|}. \quad (19)$$

The graph of the surface of the mixed derivative (19) in the vicinity of the origin (right triple) is shown in Fig. 5 on the segment of the applique from 5.9 to 6.1.

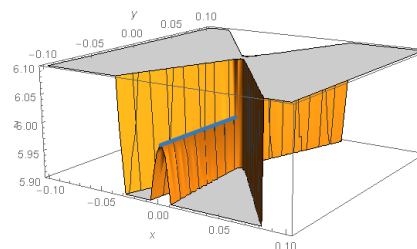


Fig. 5. The surface of the mixed derivative of the function at z from 5,9 to 6,1

The graph of the surface of the mixed derivative (19) in the vicinity of the origin (left triple) is shown in Fig. 6 on the segment of the applique from 11.9 to 12.1.

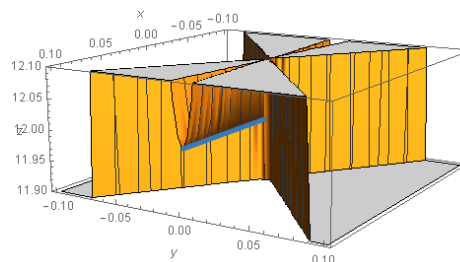


Fig. 6. The surface of the mixed derivative of the function at z from 5,9 to 6,1

3 Results and Discussion

We have proved that the mixed derivatives (19) coincide outside the point (0,0), but it is impossible to depict their behavior due to the discontinuity of the second order at the origin. However, the values and behavior in the vicinity of this point can be seen in Fig. 7. In addition, even without introducing elements of augmented reality, we can see directions where the values of mixed derivatives are 6 (abscissa), 12 (ordinate) and these numbers are equal to the values of the limits (17,18) at $a = 2$ and $b = 1$. Moreover, Fig. 7 shows directions (straight lines), where the limits of derivatives (19) have maximum values at the point (0,0). In addition, the behavior of mixed derivatives on the line $ax + by = 0$ for $a = 2$ and $b = 1$ can be seen in Fig.7.

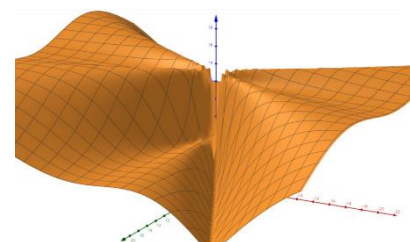


Fig. 7. The surface of the mixed derivative of the function at z from 5,9 to 12,1

Thus, under $|a| > 0$, $|b| > 0$, $|a| \neq |b|$ the analysis gives the case when the mixed second derivatives exist everywhere but do not coincide at the origin. This means that the violation of the continuity condition for mixed derivatives in the Schwartz theorem can lead to their noncoincidence. For $a = 2$, $b = 1$, the analysis of function (2) and its derivatives accompanied by visualization was carried out (Fig. 1–7) for the Schwartz theorem. Obviously, if conditions (18) are violated, i.e.: either $a = 0$, or $b = 0$, or $|a| = |b| \neq 0$, from (15–17) the equality of the mixed derivatives follows everywhere. The latter means that Schwarz theorem is not necessary. Function (2) has a line of removable discontinuities of the mixed derivative. This distinguishes it from other examples in which the discontinuity is at one point.

The calculation of derivatives of modules (6.7) simplifies the calculation by passing from a continuous to discontinuous function, but with a removable discontinuity. This is not difficult to prove. In Fig. 7, these points are a “break” line on the straight line $ax + by = 0$ ($a = 2$, $b = 1$) on the coordinate plane passing through the second and fourth quadrants (the x -axis is on the right, on the left y -axis). The values of the second mixed derivatives at the points of this line outside the point $(0,0)$ are equal to zero after the extension. Using the transition to the polar setting for the derivative (19), we obtain the graph in Fig. 8 showing the behavior of the function and the limit value segment when calculating the limits of the mixed derivative at the origin. The smallest period of the limit value function is equal to π . The analytical calculation of the extrema of this function turned out to be not an easy task.

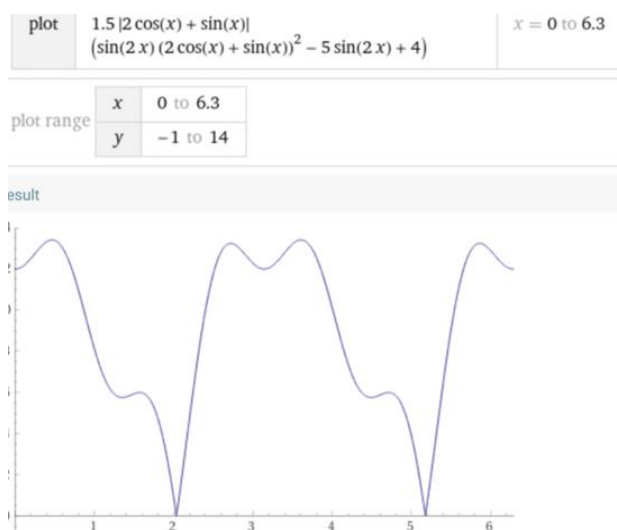


Fig. 8. Graph of the limit point function of a mixed derivative.

Present the result of the calculations.

$$\min\{ 1.5|2 \cos(x) + \sin(x)| (\sin(2x) (2 \cos(x) + \sin(x))^2 - 5 \sin(2x) + 4)\} = 0;$$

$$\begin{aligned} & \text{at } x = 2 \operatorname{arctg}\left(\frac{1}{2}(1 - \sqrt{5})\right) + 2\pi n. \\ & \max\{ 1.5 |2 \cos(x) + \sin(x)| (\sin(2x) (2 \cos(x) \\ & + \sin(x))^2 - 5 \sin(2x) + 4)\} = 6\sqrt{5}; \\ & \text{at } x = -2 \operatorname{arctg}(2 + \sqrt{5}) + 2\pi n. \end{aligned}$$

4 Conclusion

This article describes one of the methods for selecting young people for training scientific personnel. Moreover, using feasible scientific tasks, one can simultaneously improve research skills in working with analytical calculations and the ability to use SMP in science [6,9,10] and in transport [7,8]. Through the performance of scientific tasks, the student develops skills and abilities to perform complex mathematical operations and represent spatial mathematical objects on the plane (PC screen) using graphic programs and new concepts: virtuality, augmented reality, mixed reality. These concepts are necessary for the selection and construction of a flat image, which makes it possible to obtain an unambiguous perception of real and virtual three-dimensional objects. This approach to the presentation of information and its image is being developed wherever information comes in virtual form to computer screens, in particular, in maritime transport [3-5].

References

1. G.M. Fikhtengolts, *Course of Differential and Integral Calculus*: In three volumes. Volume 1 (Lan Publishing House, St. Petersburg, 1997)
2. G.A. Zelenkov, E.V. Mazanko, The use of computer technology for mathematical research with students, *Bulletin of the State Medical University named after F.F. Ushakov* **2(39)** (2022)
3. A.N. Popov, G.A. Zelenkov, D.S. Papulov, Reconstruction of various navigational scenarios of the “Ever Given” ship, including grounding in the Suez Canal using the bridge simulator with up-to-date electronic navigation charts, *Journal of Physics: Conference Series* **2061**, 012114 (2021) DOI: 10.1088/1742-6596/2061/1/012114
4. A.N. Popov, Analysis of accidents in maritime transport in the process of decision-making by shipmasters in various information environments, *Marine Intelligent Technologies* **4(46)**, 83–90 (2019)
5. E.V. Khekert, A.N. Popov, A.I. Kondratiev, D.E. Studenikin, E-Navigation mixed reality interface, *20th Annual General Assembly (AGA) of the International Association of Maritime Universities (IAMU)*, (30 October – 1 November, Tokyo, Japan, 2019) pp 40–41 (2019)
6. I.P. Boychuk, S.N. Larkov, Generator of Regular Orthogonal Grids for Numerical Solution of Gas Dynamics Problems, *Marine Intelligent Technologies* **4-3(46)**, 142–146 (2019)

7. A.V. Grinek, A.M. Fishchenko, I.P. Boychuk, D.N. Perelygin, N.V. Savosteenko, Finite element modeling of a ship generator, *Marine Intelligent Technologies*, **4-4(50)**, 92–97 (2020)
8. E.N. Syusyuka, Analysis of mathematical dependencies for calculating the trajectory of the movement of the mobile machine tool when processing ship shaft pipelines, *Operation of Sea Transport* **4(97)**, 180–185 (2020)
9. G.A. Zelenkov, M.S. Lopatin, Criteria for the study of robust stability and instability of the behavior of interval polynomials. *Marine Intelligent Technologies* **1(39(1))** (2018)
10. V.A. Bagina, S.A. Balyaeva, O.A. Borovkova, *Modern Educational Technologies* (CRNS, Novosibirsk, 2017)