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Analyzing Optimal Portfolios of Eleven Assets under Different Constraints

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Abstract. Portfolios, which allocate investor capital to different investments, are a common risk management strategy. It aims to spread investment risk using diversification. The purpose of this paper is to allocate assets to stocks in the technology sector, consumer sector, and pharmaceutical sector. Stocks of SPX500 and ten companies are selected, and the Yahoo Finance database in Python is utilized to export the historical data, and the knowledge of statistics is applied to calculate the basic values of the stocks. For example, data such as annualized average return, annualized standard deviation, and alpha. These data are utilized to obtain the correlation coefficients between eleven assets and to plot the Capital Allocation Line (hereinafter referred to as CAL), efficient frontier and inefficient frontier images under different constraints. The results show that PG's asset share is the highest among the five different constraints, both in the minimum variance case and in the maximum Sharpe ratio case. After imposing constraints, the portfolio's return at the same risk decreases.

1 Introduction

In 1952, Markowitz first proposed portfolio theory. The theory mainly consists of the mean-variance model and the efficient boundary of a portfolio. He proposed a theoretical framework for diversification to reduce overall investment risk. The principle is to utilize the correlation between different assets for asset allocation [1]. When an investor chooses a portfolio, one's will bound to choose the one that minimizes risk and maximizes return. In other words, investors need both more return and accept affordable risk [2]. Therefore, how to weigh risk and return is an unchanging topic in risky investment.

As of today, there are many studies on portfolios. However, most of them are portfolio studies for a single sector. For example, Sen and Dutta designed mean-variance optimized portfolios for six important sectors [3]. Sen, Mondal and Mehtab analyzed the optimal risky portfolios for five sectors using LSTM models [4]. Keating's study for a single real estate investment [5]. In addition, there are some studies for portfolios under certain constraints. For example, Grauer and Shen compare the difference between portfolios with and without constraints [6]. Behr and Guettler explore a constraint that also explores the minimum variance, but the portfolio strategy under this constraint reduces the risk and increases the return [7].

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In this article, ten stocks from three different sectors, technology sector, consumer sector and pharmaceutical sector, will be selected and formed into a portfolio with SPX500. The idea of Markowitz model is utilized to study the optimal asset allocation of these eleven assets. First, the initial data from September 1, 2014, to September 1, 2024, is exported from Yahoo Finance's database using a Python program. Then, the data at the end of each month is selected as the calculation data. Next, the data of annualized mean return, annualized standard deviation, alpha, beta and residual standard deviation were calculated in Excel using statistical methods. After that, these data are utilized to calculate the correlation coefficients between two assets. What's more, the Markowitz model is built under five different constraints to obtain the minimum variance portfolio, the maximum Sharpe ratio portfolio, draw the capital allocation line, the minimum variance frontier, the efficient frontier and the inefficient frontier. Finally, analyze the trend of each asset's allocation and risk and return under these five constraints.

The structure of this article is consistent with the above steps: the second part is the data, the third part is the methodology, the fourth part is the results, and the fifth part is the conclusion.

2 Data

The data for this post came from Yahoo finance's database in Python. After using the Python language, it was convenience to export the historical data of the selected stock. This article selects the SPX500 and ten different stocks. Specifically, this article uses the closing prices of ten stocks, AAPL, MSFT, GOOGL, AMZN, TSLA, NVDA, JNJ, INTC, PG and CL, from August 30, 2014, to August 30, 2024. The stocks selected for this article cover the technology, consumer goods and pharmaceuticals sectors, and they are all leaders in these industries. The purpose is that they are all more comprehensive and easier to analyze systematically. After a series of calculations and collations, this ten-year data is calculated to obtain the values of mean annual return, annualized standard deviation, beta, alpha, and residual standard deviation. The basic information of spx500 and 10 different stocks is shown in Table 1.

Table 1. Basic statistics of the selected assets

	'SPX'	'AAPL'	'MSFT'	'GOOGL'	'AMZN'	'TSLA'	'NVDA'	'JNJ'	'INTC'	'PG'	'CL'
Mean	9.3%	25.14%	23.78%	17.71%	26.42%	40.33%	65.88%	5.94%	0.01%	8.86%	6.06%
StDev	15.3%	28.0%	21.9%	24.0%	31.3%	61.5%	46.9%	15.6%	31.6%	15.9%	15.7%
Beta	1.000	1.248	0.997	1.032	1.260	1.807	1.792	0.577	0.990	0.414	0.509
Alpha	0.0%	13.5%	14.5%	8.1%	14.7%	23.5%	49.2%	0.6%	-9.2%	5.0%	1.3%
RS	0.0%	20.5%	15.7%	18.0%	24.6%	54.9%	38.0%	12.9%	27.7%	14.5%	13.6%

Note: RS is residual standard.

As can be seen from Table 1, NVDA has the largest average annual return and the largest standard deviation. INTC has the smallest average annual return, but its standard deviation is not low, reflecting the fact that this stock is not only less rewarding but also risky, which makes it not an ideal choice. The rest of the stocks' fundamental data do not stand out.

3 Methods

In this article, a portfolio consisting of the selected SPX500, and ten different stocks is analyzed through Markowitz's mean-variance model using knowledge related to linear algebra, probability theory, investment science, etc. Reasonable investment allocation was made under five different constraints was analyzed.

3.1 Correlation coefficient

Correlation is a statistic that measures the extent to which two securities move in relation to each other. Correlation between stocks is used in the study of portfolio management. For example, suppose the correlation coefficient between stock A and stock B is negative, then if the price of stock A falls then the price of stock B will rise; Conversely, if the correlation coefficient between stock A and stock B is positive, then the price of stock A and the price of stock B will both rise or fall. Using this principle, choosing two assets that are negatively correlated to hedge the risk or two assets that are positively correlated to get as much profit as possible but with higher risk [8]. The value of the correlation coefficient shows how the price of one stock changes when the price of another stock rises or falls. Markowitz's mean-variance model capitalizes on this by saying that if the correlation between two stocks is in the interval -1 < r < 0 where r is correlation coefficient, then when one stock's price goes down the other's price goes up to offset some of the loss, therefore reducing the overall risk of the portfolio. The formula for the correlation coefficient is shown below.

$$Corr = \frac{n * (\sum(X,Y) - (\sum(X) - \sum(Y)))}{\sqrt{(n * \sum(X^2) - \sum(x)^2) * (n * \sum(Y^2) - \sum(Y)^2)}} \tag{1}$$

Note: Corr is correlation coefficient. n is equal to Number of observations. X and Y represent the excess return of two different stocks over a 10-year period [9].

3.2 Mean-Variance model

The mean-variance model was introduced by Harry Markowitz in 1952. Its purpose is to determine the optimal portfolio by balancing the expected return of a security against the associated risk [5]. The following are the key principles of the mean-variance model.

First, the mean of the returns $\left(E(R_p)\right)$ measures the average return of such a portfolio.

$$E(R_p) = \sum_{i=1}^{11} w_i * E(R_i) \tag{2}$$

Where $E(R_i)$ is annualized average return. At this point, the risk of systems is usually expressed as the standard deviation ($\sigma(R_p)$) of the portfolio.

$$\sigma(R_p) = \sqrt{Var(R_p)} = \sqrt{Var\left(\sum_{i=1}^{11} w_i\, R_i\right)} = \sum_{\substack{0< i \le 11 \\ 0<j \le 11}} (w_i S_i) * Corr_{i,j} * (w_j S_j) \tag{3}$$

Where S_i is the standard deviation of the i-th asset (i.e., the risk of an individual asset). Furthermore, using the above data, below is the calculated Sharpe ratio and obtained Capital Allocation Line.

$$sharpe\ ratio = \frac{E(R_p) - R_f}{\sigma_p}, E(R_c) = r_f + \frac{E(R_p) - R_f}{\sigma_p} * \sigma_c \tag{4}$$

Where σ_c is the standard deviation of the portfolio (including risk-free and risky investments) and $E(R_c)$ is expected return of the portfolio. Finally, this article will also show three curves, which are the minimum variance frontier curve, the efficient frontier curve and the inefficient frontier curve. The curve that connects the points of the minimum level of investment risk that can be realized at different returns is the minimum variance frontier. Given a level of risk, there is always a portfolio with the highest expected return. These portfolios are called efficient portfolios. The curve formed by these efficient portfolios is called the efficient frontier. It's also the upper half of the minimum variance frontier. Then, the lower half of the minimum variance frontier is the inefficient frontier [10].

4 Results

First, the value of the correlation coefficient between the two assets needs to be obtained. The method used in the article is to place the data in Excel and then set up the function CORREL () to solve for the correlation coefficients between the SPX 500 and the ten stocks, which ultimately results in the Table 2 shown below.

Table 2. Correlation coefficient

	SPX	AAPL	MSFT	GOOGL	AMZN	TSLA	NVDA	JNJ	INTC	PG	CL
SPX	1.000	0.682	0.697	0.660	0.617	0.450	0.585	0.565	0.480	0.400	0.496
AAPL	0.682	1.000	0.582	0.467	0.535	0.523	0.542	0.360	0.360	0.290	0.275
MSFT	0.697	0.582	1.000	0.634	0.612	0.370	0.584	0.362	0.490	0.300	0.310
GOOGL	0.660	0.467	0.634	1.000	0.629	0.351	0.504	0.209	0.359	0.165	0.272
AMZN	0.617	0.535	0.612	0.629	1.000	0.426	0.537	0.204	0.327	0.011	0.203
TSLA	0.450	0.523	0.370	0.351	0.426	1.000	0.329	0.212	0.225	0.016	-0.021
NVDA	0.585	0.542	0.584	0.504	0.537	0.329	1.000	0.104	0.371	0.005	0.041
JNJ	0.565	0.360	0.362	0.209	0.204	0.212	0.104	1.000	0.315	0.475	0.525
INTC	0.480	0.360	0.490	0.359	0.327	0.225	0.371	0.315	1.000	0.110	0.075
PG	0.400	0.290	0.300	0.165	0.011	0.016	0.005	0.475	0.110	1.000	0.658
CL	0.496	0.275	0.310	0.272	0.203	-0.021	0.041	0.525	0.075	0.658	1.000

From the above table it can be concluded that the highest value of correlation coefficient between MSFT and SPX is 0. 697. The minimum correlation coefficient between TSLA and CL is -0.021.

Then, in this section of the article, a calculation will be made of the point at which the maximum Sharpe ratio is reached, and four curves will be drawn: the Capital Allocation Line, the Minimum Variance Frontier, the Efficient Frontier and the Inefficient Frontier. There are five different constraints here. This article will accomplish the above purpose under different constraints.

Constraint 1: The sum of the absolute values of the percentages of all asset weights is less than 2, i.e. $\sum_{i=1}^{11} |w_i| \leq 2$. Its results are shown in Table 3 and Fig. 1.

Table 3. Percentage of each asset under the constraint 1

	SPX	AAPL	MSFT	GOOGL	AMZN	TSLA	NVDA	JNJ	INTC	PG	CL
minVar	0.30	-0.08	-0.02	0.06	0.04	0.00	0.00	0.24	0.03	0.30	0.14
maxsharpe	-0.26	0.00	0.25	0.00	0.05	0.05	0.41	0.17	-0.24	0.56	0.00

Under the Maximum Sharpe Ratio, PG has the highest percentage of assets at 0.56. Some of these assets have negative weights. For example, SPX has a weight of -0.26 and INTC has a weight of -0.24, which means that these assets are shorted. Maximum Sharpe ratio implies maximizing profits and hence specific assets are utilized to optimize the performance of the portfolio, so overall the weights of the assets vary widely. GOOGL and CL have a weight of 0, which means that they are not considered in the asset allocation. With minimum variance, the asset share of SPX and PG is at most 0.3. In this case, there are also still some assets with negative weights. For example, AAPL has a weight of -0.08 and MSFP has a weight of -0.02, which means that these assets are shorted. However, compared to the weights calculated at the maximum Sharpe ratio, these shorted assets do not account for a high percentage of the assets. TSLA and NVDA have a weight of 0, suggesting that they are not considered in the asset allocation.

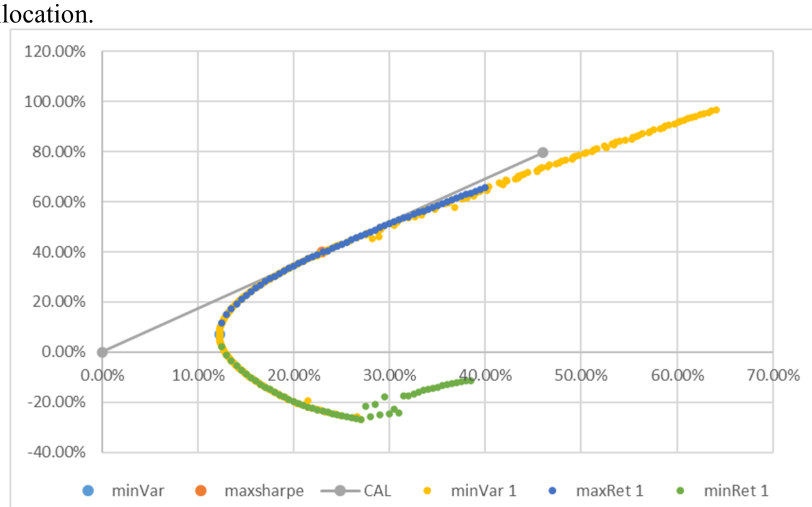

Fig. 1. Minimum-variance frontier and capital allocation line under the constraint 1

Notes: minVar for minimum variance; maxSharpe for maximum Sharpe ratio. The point at the tip of the minimum-variance frontier is the point at which the global minimum variance is also the point at which the maximum Sharpe ratio is. The pink line is the capital allocation line. The part in purple is the Efficient Frontier. The part in green is the Inefficient Frontier.

Constraint 2: The absolute value of the percentage of each asset is less than 1, i.e. $|w_i| \leq 1$. Its results are shown in Table 4 and Fig. 2.

Table 4. Percentage of each asset under the constraint 2

	^SPX	AAPL	MSFT	GOOGL	AMZN	TSLA	NVDA	JNJ	INTC	PG	CL
minVar	0.30	-0.08	-0.02	0.06	0.04	0.00	0.00	0.24	0.03	0.30	0.14
maxsharpe	-1.00	-0.02	0.39	0.04	0.12	0.08	0.53	0.42	-0.34	0.71	0.07

Note: minVar is minimum variance.

Under the Maximum Sharpe Ratio, PG has the highest percentage of assets at 0.71. Some of these assets have negative weights. For example, SPX has a weight of -1, INTC has a weight of -0.34, and AAPL has a weight of -0.02. As in Constraint 1, this means that these assets are shorted. No asset has a weight of 0. That is, all assets are considered. With minimum variance, the values are the same as in Constraint 1.

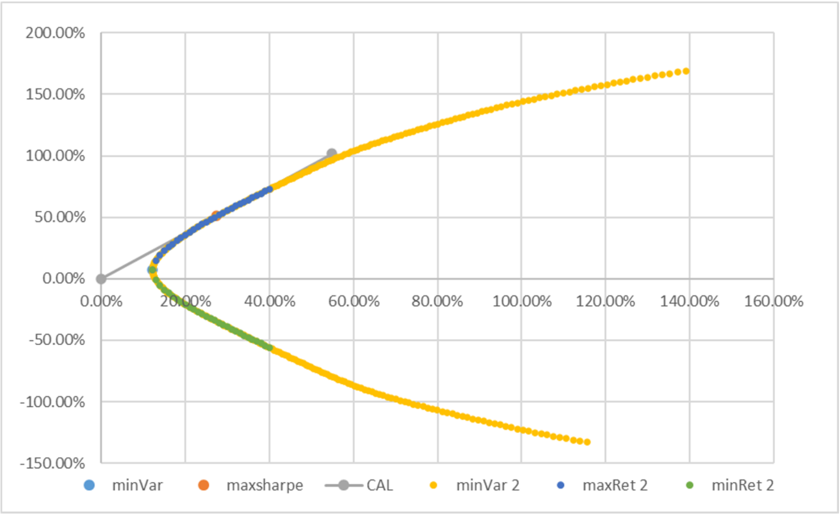

Fig. 2. Minimum-variance frontier and capital allocation line under the constraint 2

Notes: minVar for minimum variance; maxSharpe for maximum Sharpe ratio. The point at the tip of the minimum-variance frontier is the point at which the global minimum variance is also the point at which the maximum Sharpe ratio is. The pink line is the capital allocation line. The part in purple is the Efficient Frontier. The part in green is the Inefficient Frontier.

Constraint 3: The simplest constraint in the investment decision is the absence of a specific restriction. The results are shown in Table 5 and Fig. 3.

Table 5. Percentage of each asset under the constraint 3

	SPX	AAPL	MSFT	GOOGL	AMZN	TSLA	NVDA	JNJ	INTC	PG	CL
minVar	0.30	-0.08	-0.02	0.06	0.04	0.00	0.00	0.24	0.03	0.30	0.14
maxsharpe	-2.63	0.08	0.60	0.19	0.22	0.15	0.84	0.76	-0.45	0.98	0.26

Note: minVar is minimum variance.

Under the maximum Sharpe ratio, PG has the highest asset share of 0.98. Some of these assets have negative weights. For example, SPX has a weight of -2.63 and INTC has a weight of -0.45. As in Constraint 1, this means that these assets are shorted. No asset has a weight of 0. That is, all assets are considered. With minimum variance, the values are the same as in Constraint 1.

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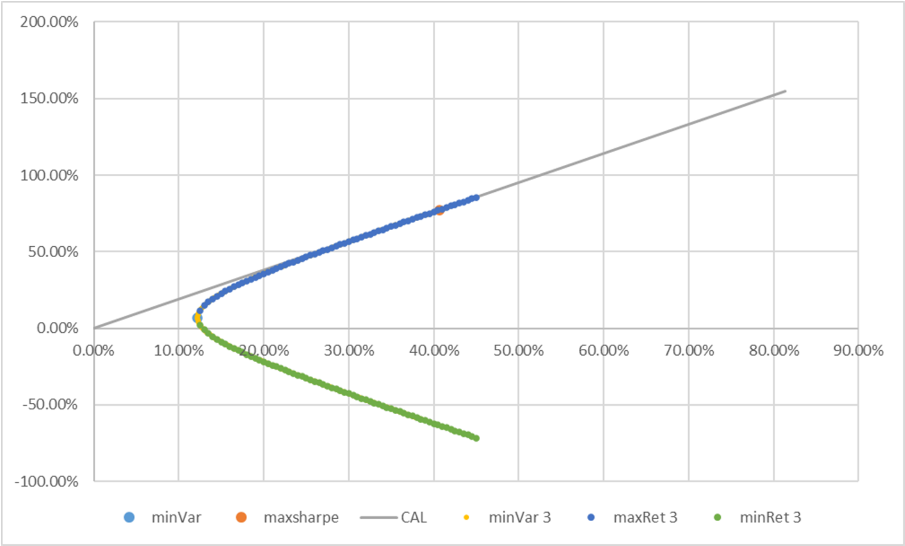

Fig. 3. Minimum-variance frontier and capital allocation line under the constraint 3

Notes: minVar for minimum variance; maxSharpe for maximum Sharpe ratio. The point at the tip of the minimum-variance frontier is the point at which the global minimum variance is also the point at which the maximum Sharpe ratio is. The pink line is the capital allocation line. The part in purple is the Efficient Frontier. The part in green is the Inefficient Frontier.

Constraint 4: The ratio of each asset is greater than 0, i.e.$W_i \geq 0$. That is, shorting is not allowed. This situation greatly limits the amount of risk. The results are shown in Table 6 and Fig. 4.

Table 6. Percentage of each asset under the constraint 4

	SPX	AAPL	MSFT	GOOGL	AMZN	TSLA	NVDA	JNJ	INTC	PG	CL
minVar	0.20	0.00	0.00	0.07	0.01	0.00	0.00	0.25	0.02	0.28	0.17
maxsharpe	0.00	0.00	0.08	0.00	0.00	0.04	0.40	0.00	0.00	0.48	0.00

Note: minVar is minimum variance.

Under the maximum Sharpe ratio, PG has the highest asset share of 0.48. Because of the constraints, no asset has a negative share. That is, no assets are allowed to be shorted. TSLA, MSFT and NVDA have a share of 0.04, 0.08 and 0.4 respectively. The rest of the assets have a share of zero. With minimum variance, PG has the highest asset share of 0.28. Similarly, no asset has a negative share. Among them, AAPL, MSFT, NVDA and TSLA have zero share. Thus, these assets are not considered.

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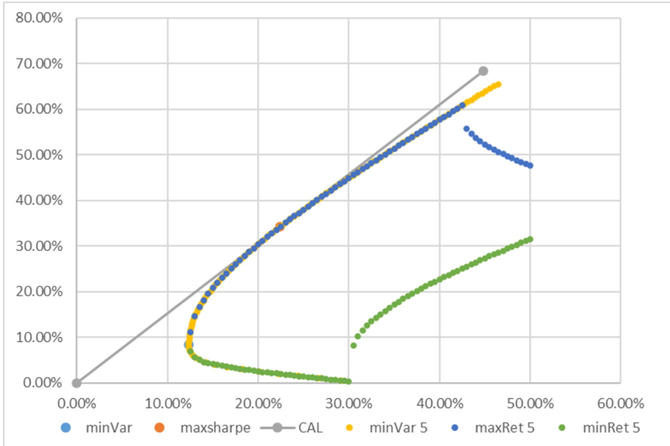

Fig. 4. Minimum-variance frontier and capital allocation line under the constraint 4

Notes: minVar for minimum variance; maxSharpe for maximum Sharpe ratio. The point at the tip of the minimum-variance frontier is the point at which the global minimum variance is also the point at which the maximum Sharpe ratio is. The pink line is the capital allocation line. The part in purple is the Efficient Frontier. The part in green is the Inefficient Frontier.

Constraint 5: The first asset share is 0, i.e. $W_1 = 0$. This constraint is mainly favored by investors who need to avoid the impact of a particular asset. The results are shown in Table 7 and Fig. 5.

Table 7. Percentage of each asset under the constraint 5

	SPX	AAPL	MSFT	GOOGL	AMZN	TSLA	NVDA	JNJ	INTC	PG	CL
minVar	0.00	-0.06	-0.01	0.11	0.05	0.00	0.02	0.32	0.04	0.33	0.20
maxsharpe	0.00	-0.10	0.42	-0.13	0.07	0.06	0.53	0.14	-0.47	0.66	-0.19

Note: minVar is minimum variance.

Under the maximum Sharpe ratio, PG's share of assets is the highest at 0.66 and INTC's share is the lowest at -0.47. Because of the constraints, the share of SPX is 0. Under minimum variance, PG's share of assets is highest at 0.33 and AAPL's share is lowest at -0.06. Similarly, SPX's share is 0. Apart from that, TSLC's share is 0.

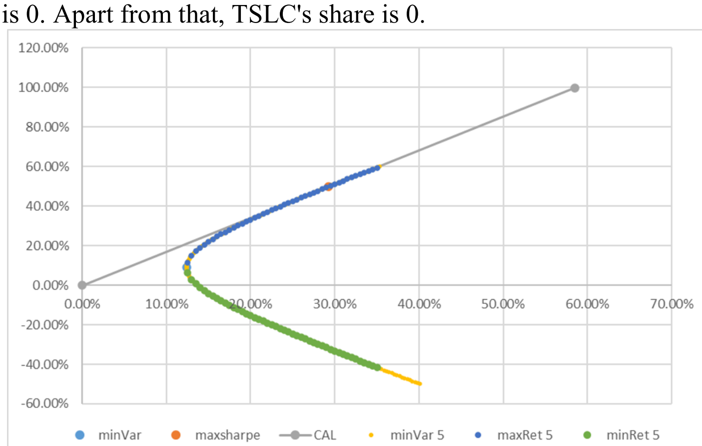

Fig. 5. Minimum-variance frontier and capital allocation line under the constraint 5

Notes: minVar for minimum variance; maxSharpe for maximum Sharpe ratio. The point at the tip of the minimum-variance frontier is the point at which the global minimum variance is also the point at which the maximum Sharpe ratio is. The pink line is the capital allocation line. The part in purple is the Efficient Frontier. The part in green is the Inefficient Frontier.

5 Conclusion

In this paper, starting with the database of Yahoo Finance in Python to derive the historical data of selected SPX and 10 stocks. Then, the data is analyzed and organized into data charts using the knowledge of statistics. Basic statistics and correlation coefficients between the 11 assets are included here. Secondly, the mean-variance model is applied to obtain the portfolio at minimum variance and maximum Sharpe ratio respectively and the images of Capital Allocation Line, efficient frontier and inefficient frontier are plotted under one of the constraints. It is found that at the same variance, i.e., at the same risk, they all have lower returns compared to the unconstrained condition. In other words, all constraints except the unconstrained condition reduce returns increase risk. Investors should be clear about the impact of constraints, and consider choosing the cut-off point of the CAL from the efficient frontier, as this point is the optimal risk portfolio available to the investor.

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