Research on insurance pricing under option game based on Black-Scholes model

Yicheng Zhang[1*]

[1]China University of Petroleum (Beijing), Karamay Campus, Beijing, 834000, China

Abstract. The pricing of insurance products has always occupied a central position in the insurance business and has long been an important focus of academic research. With the development of theory, the combination of option theory and game theory provides a new analytical perspective for insurance pricing. Specifically, insurance can be viewed as a kind of option so that the strategies in option game theory can be applied to determine a reasonable premium level. In this context, this study aims to explore in depth how option game theory can be applied to the insurance pricing problem. First, this paper defines insurance as a special form of option and tries to borrow the classical option pricing model, i.e., the Black-Scholes model, to explore the insurance pricing problem in this new perspective. The Black-Scholes model is an important tool in the field of option pricing, through which a reasonable insurance pricing approach can be derived. Secondly, this paper further introduces the concept of game theory and constructs a basic framework of games in insurance pricing. In this framework, the interaction between insurance companies and policyholders is regarded as a game process, and the decision-making behavior of both parties directly affects the price formation of insurance. Game theory provides a new theoretical basis for understanding the pricing mechanism in the insurance market. Finally, this paper focuses on the potential application of American put options in insurance pricing based on option game theory.

1 Introduction

The insurance industry originated from the compensation for unexpected events with small probability. With the development of social diversification, it gradually covers the safety of personal life and property, the insurance safety system of enterprises and the country, and has a profound impact on the whole society[1]. In the early stage of development of the country's insurance industry, there is lack of local theory, so this paper mainly learns from foreign actuarial theory. Insurance economics has provided a research foundation in this process and has developed together with finance[2].

With the development of the economy and the increasingly close connection of the global economy, financial derivatives are becoming more and more powerful in China. More and more Chinese researchers use financial derivatives to solve financial problems and meet the needs of the public. As an important financial derivative product, option emerged in the 1970s

Corresponding author: WMORR35485@my.sunywcc.edu

because of its complete pricing system theory, which provides ideas for solving the pricing problems in actuarial science. Nowadays, options play an important role in China's financial investment market and are widely used in the insurance industry [3].

Option game is based on the mature development of game theory and option theory, combined with the rich pricing theory of option and the strategic thinking and modeling characteristics of game [4]. Based on the theory of insurance economics, option pricing theory and game theory, this paper uses the method of option game to study the pricing of partial insurance. It is hoped that through discussion, this paper can find an insurance pricing scheme that is more in line with real life, so as to maximize the common interests of insurers and policyholders and achieve the purpose of fairness.

At present, the theory of option game mainly studies the part of real option, considering the time value and investment decision of the project [5]. For example, Huisman's study on the entry strategy of two enterprises under the competition of new technology R&D under the uncertain success rate [6]. Trigeorgis' choice of the optimal start time of the project under competition [7]. Grenadier's research on the optimal exercise time of investment options in duopoly markets in the real estate industry [8]. Agurerrevere's game selection strategy about the quality of non-stored goods and investment time, etc. [9].

In recent years, there have also been applications of option game in insurance pricing. For example, in the pricing process of deductible insurance in option game, Sun divided the amount of insurer's compensation into two categories according to the definition of deductible insurance: No compensation below the deductible and compensation above the deductible, with the compensation method as the boundary condition, determine the appropriate insurance price through the game between the two parties [10]. In reality, this classification of compensation amount is too broad, which cannot reach the ideal insurance price of both parties. On this basis, the model in this chapter is further subdivided by combining the idea of proportional reinsurance, setting a threshold value of compensation amount, compensating all the part below the threshold value, and dividing the compensation amount exceeding the threshold value according to the proportion. Losses are again shared between the insurer and the policyholder, which in turn determines the price of a more appropriate part of the insurance [11, 12].

2 Model construction

2.1 Hypotheses of the model

This paper assumes the following qualifications: Economic activities are carried out under perfect information and are risk-free probability measures; There are no risk-free arbitrage opportunities; There are no transaction costs, short sale restrictions and taxes in the process of option trading.

Here are the meanings of the parameters in the model: V denotes the initial pricing of the subject matter insured; S denotes the cut-off value of the insurance company's compensation, Π represents the insurer's expected underwriting profit, $V(t)$ represents the value of the underlying asset t at the moment, and let $V(t)$ follow the geometric Brownian motion, that is,

$$dV(t) = \rho V(t)dt + \sigma V(t)dZ \quad (1)$$

Where ρ represents the drift rate of Brownian motion, σ denotes the volatility of the Brownian motion, dZ denotes the change in the standard Brownian motion. Drift rate and volatility are set as constants in this model process.

 https://doi.org/10.1051/shsconf/202420803005

2.2 Option game analysis

The following is a detailed analysis of the strategy of the insurer and the policyholder to maximize the insurance price in line with their own interests in the process of setting the insurance price, but within a range acceptable to the other trader [11].

First of all, this part of the insurance is regarded as an American put option purchased by the policyholder at the insurer: the value of the subject matter insured is the price of the subject matter in the option; The money that policyholder buys insurance is equivalent to the cost of the first option, that is, the option price; No matter when the insured person's insurance subject matter loss, he can claim compensation, it is the American option can be executed at any time characteristics. When the policyholder according to the agreement, in $V_0 - V > S$; Getting compensation $a(V_0 - V - S) + S$ is equivalent to exercising the American put option at a strike price of $V_0 - V$.

After establishing the option identity of insurance, the following uses the method of option game to analyze what strategies are adopted by the two players (the insurer and the policyholder) to finally determine the maximum interests of each purpose (in line with the perfect minimum insurance price in each other's mind) in the process of change:

First, determine the game strategy of the players in the game. For an option (insurance contract) with a long duration of T, the option buyer (policyholder) hopes to have an optimal price value of the subject matter within $\tau = T - t$ when the option expires, so as to achieve the highest value of the option $P(V, S - V, \tau)$; For the seller of the option (the insurer), he hopes to have the most appropriate deductible S and the proportion adjustment factor $\alpha \in [0,1]$ to maximize the interest in the option trading (the insurer's maximum expected underwriting profit). In the process of the game, the insurer takes the first action to determine the deductible amount S and the proportion α ; After that, under complete information, after the policyholder understands the insurer's strategic plan, the policyholder determines the optimal value of the insurance subject matter $\tilde{V}(t)$.

Secondly, this paper uses the relevant knowledge of options and the classical model of option pricing to study the partial risk model of this paper.

In the partial insurance of this paper, the option value satisfies the Black-Scholes equation, that is,

$$\frac{1}{2}\sigma^2 V^2 P_{VV} + rVP_V + P_t - rP = 0 \qquad (2)$$

Where r is the risk-free interest rate, and the boundary conditions are as follows:

$$\boldsymbol{P}(\infty) = \boldsymbol{0} \qquad (3)$$

$$P\left(\tilde{V}(t)\right) = \begin{cases} a\left(V_0 - \tilde{V} - S\right) + S, V_0 - \tilde{V} > S \\ V_0 - \tilde{V}, V_0 - \tilde{V} \le S \end{cases} \qquad (4)$$

$$P(V, V_0 - S, \tau) \ge Max\left(0, P\left(V(t)\right)\right) \qquad (5)$$

3 Results and discussion

3.1 Solution process

The following is to find the optimal [illegible] and S^* through the Black-Scholes equation as well as the boundary conditions. Firstly, let's analyze Equation (2), which is a Black-Scholes equation of American put option, so it cannot be directly solved for its analytical solution. In this case, the American put option is regarded as a permanent option (i.e., $T \to \infty$), and the analytical solution of the equation can be calculated by mathematical methods. When $T \to \infty$, Equation (2) becomes:

$$\frac{1}{2}\sigma^2 V^2 P_{VV} + rVP_V - rP = 0 \quad (6)$$

This is a second-order linear differential equation with variable coefficients, with polynomials with coefficients V and quadratic, primary, and zeroth order. Try to take the solution below:

$$P(V) = \alpha V^2 + \beta V + \gamma \quad (7)$$

Then bring $P_V = 2\alpha V + \beta, P_{VV} = 2\alpha$ into the equation (6) as,

$$\frac{1}{2}\sigma^2 V^2 \cdot 2\alpha + rV(2\alpha V + \beta) - r(\alpha V^2 + \beta V + \gamma) = 0 \quad (8)$$

It can be collapsed to get

$$(\sigma^2\alpha + r\alpha)V^2 - r\gamma = 0 \quad (9)$$

Compare the same power coefficients below to get the equation $\sigma^2\alpha + r\alpha = 0, r\gamma = 0$. This leads to $\alpha = 0, \gamma = 0$, a special solution of the equation is obtained when $\beta = 1$:

$$P_1(V) = V \quad (10)$$

One can also take a solution of the form:

$$P(V) = V^m (m \neq 1) \quad (11)$$

Substituting $P_V = mV^{m-1}, P_{VV} = m(m-1)V^{m-2}$ into Eq.(6) gives: $\left(\frac{1}{2}\sigma^2 m + r\right)(m-1) = 0$, thus, $\frac{1}{2}\sigma^2 m + r = 0, m = -\frac{2r}{\sigma^2}$. This leads to another special solution of equation (6):

$$P_2(V) = V^{-\frac{2r}{\sigma^2}} \quad (12)$$

And it is easy to know that $P_1(V)$ and $P_2(V)$ are linearly independent, so the general solution of the second order linear variable coefficient differential equation (6) is:

$$P(V) = C_1 V + C_2 V^{-\eta}, \ \eta = \frac{2r}{\sigma^2} \quad (13)$$

The values of C_1 and C_2 are discussed below in the light of the boundary conditions. From the boundary condition $P(\infty) = 0$ the author get $C_1 = 0$, at this point

$$P(V) = C_2 V^{-\eta} \quad (14)$$

Continue the solution according to the boundary condition (4). When $V_0 - \tilde{V} > S$, there is $P(\tilde{V}) = a(V_0 - \tilde{V} - S) + S = C_2 \tilde{V}^{-\eta}$. Thus,

$$C_2 = [a(V_0 - V - S) + S]\tilde{V}^{\eta} \quad (15)$$

Substituting equation (15) into equation (14) gives:

$$P(V) = \left[a\left(V_0 - \tilde{V} - S\right) + S\right](\frac{V}{\tilde{V}})^{-\eta} \quad (16)$$

3.2 Policyholder's perspective

From the perspective of the insured, the optimal $\tilde{V}$ is utilized to obtain the maximum $P(\tilde{V})$ value by applying a partial derivative to $\tilde{V}$ using equation (16) and making the partial derivative equal to 0 can obtain $\left[a\frac{\eta}{\tilde{V}}(V_0 - S) + \frac{\eta}{\tilde{V}}S - a\eta - a\right](\frac{V}{\tilde{V}})^{-\eta} = 0$. Thus,

$$\tilde{V} = \frac{[a(V_0 - S) + S]\eta}{a + a\eta} \quad (17)$$

At this point it is necessary to have $V_0 - \tilde{V} - S = V_0 - \frac{[a(V_0-S)+S]\eta}{a+a\eta} - S = \frac{a(V_0-S)-S\eta}{a+a\eta} > 0$. Due to $\eta = \frac{2r}{\sigma^2} > 0, a > 0$. To make $V_0 - \tilde{V} > S$, i.e., $V_0 - \tilde{V} - S > 0$, it is only necessary to have $a(V_0 - S) - S\eta > 0$, i.e., to satisfy the equation $S < \frac{aV_0}{a+\eta}$. Substituting (17) into (16) yields the maximum option value at this point: $P(V) = \frac{aV_0+(1+\eta)S}{1+\eta}(\frac{V(a+a\eta)}{[a(V_0-S)+S]\eta})^{-\eta}$, Where $S < \frac{aV_0}{a+\eta}$. When $V_0 - \tilde{V} \leq S$, there is $P(\tilde{V}) = V_0 - \tilde{V} = C_2\tilde{V}^{-\eta}$. Thus

$$C_2 = (V_0 - \tilde{V})\tilde{V}^{\eta} \tag{18}$$

Substituting equation (18) into equation (14) gives:

$$P(V, \tilde{V}) = (V_0 - \tilde{V})\left(\frac{V}{\tilde{V}}\right)^{-\eta} \tag{19}$$

As with the analysis above, the partial derivation of V using (19) is obtained here: $\frac{\partial P}{\partial \tilde{V}} = \left[\frac{V_0\eta}{\tilde{V}} - (1+\eta)\right]\left(\frac{V}{\tilde{V}}\right)^{-\eta}$. When $\frac{\partial P}{\partial \tilde{V}} = 0$, there is

$$\tilde{V} = \frac{V_0\eta}{1+\eta} \tag{20}$$

At this point there is $V_0 - \tilde{V} - S = V_0 - \frac{V_0\eta}{1+\eta} - S = \frac{V_0}{1+\eta} - S$, to satisfy $V_0 - \tilde{V} \leq S$, only $S \geq \frac{V_0}{1+\eta}$ is required.

Therefore, the optimal option value is obtained by substituting equation (20) into (19) at this point: $P(V) = \frac{V_0}{1+\eta}[\frac{V(1+\eta)}{V_0\eta}]^{-\eta}$, Where $S \geq \frac{V_0}{1+\eta}$.

3.3 Insurer's perspective

The following considerations are made from the perspective of the insurer, who finds the appropriate indemnity amount fraction S to maximize the expected underwriting profit, and whose decision is based on the insured's choice of $\tilde{V}$. When $S < \frac{aV_0}{a+\eta}$, there is $P(V,S) = \frac{aV_0+(1+\eta)S}{1+\eta}\left(\frac{V(a+a\eta)}{[a(V_0-S)+S]\eta}\right)^{-\eta}$. The insurer's maximum expected underwriting profit is:

$$Max\Pi = P(V,S) - \int_0^{V_0-S} e^{-rt}\,[a(V_0 - V - S) + S]f(V)dV \tag{21}$$

The following analysis of the formula (21), the formula $f(V)$ for the density function of V, by the problem that V is subject to geometric Brownian motion, so it is itself a lognormal distribution of random variables, the density function can be sought; there is an expectation of payout of the discount term, but the time t for the benefit of the random term, more difficult to compute in practice, so this formula to take the risk-free interest rate $r = 0$ to simplify the arithmetic, the formula becomes:

$$Max\Pi = P(V,S) - \int_0^{V_0-S}[a(V_0 - V - S) + S]\,f(V)dV \tag{22}$$

Using (22) the first order derivative of S and making it equal to 0 yields:

$$\frac{\partial P}{\partial S} - \int_0^{V_0-S}(1-a)\,f(V) + Sf(V_0 - S) = 0 \tag{23}$$

At infinity (23) changes to:

$$\left[\frac{\eta}{1+\eta}\cdot\frac{aV_0+(1+\eta)S}{aV_0+(1-a)S}+1\right]\left(\frac{V(a+a\eta)}{[a(V_0-S)+S]\eta}\right)^{-\eta}-\int_0^{V_0-S}(1-a)\,f(V)+Sf(V_0-S) \quad (24)$$

This can be solved to obtain the appropriate S^*, thus have the optimal premium $P(S^*)$. When $S\geq\frac{V_0}{1+\eta}$, there is $P(V)=\frac{V_0}{1+\eta}\left[\frac{V(1+\eta)}{V_0\eta}\right]^{-\eta}$. The maximum expected underwriting profit is:

$$Max\Pi=P(V,S)-\int_{V_0-S}^{+\infty}V_0-Vf(V)dV \quad (25)$$

The first order partial derivative with respect to S is obtained by making it equal to 0:

$$Sf(V_0-S)=0 \quad (26)$$

It is also possible to obtain that the appropriate $S^*>0$ at this point, thus have an optimal premium $P(S^*)$.

4 Conclusion

In this paper, the author proposes an insurance pricing model by treating insurance as an American put option and using a combination of option theory and game theory. In the process of research, the Black-Scholes model is first introduced as the basis of pricing, and a game framework is constructed on this basis to analyze how the insurer and the policyholder maximize their interests through the game under the condition of complete market. The core of this paper is to propose an optimal pricing model that can satisfy the interests of both the insurer and the policyholder, especially in the framework of American put option, the specific insurance price and cut-off amount are obtained.

The method of option game is used to study insurance pricing, which has changed the situation that only insurance companies set the standard premiums alone in the past. It is to reach a consensus between the insurer and the policyholder through mutual strategies, and then develop a perfect method that meets the expected interests of both the insurer and the policyholder. Although there has been an option game pricing model for deductible insurance, in this model, the formulation of deductible is used to adopt a strategy to obtain a fair insurance price. In this paper, on the basis of deductible and proportional insurance, the coefficient a is introduced to divide the liability of compensation above the threshold S, so as to adapt to the changes of economic situation more flexibly. Also more consistent with the implications of no-arbitrage prices.

In this paper, the author only makes a theoretical exploration when using the option game method to study the insurance pricing problem. Some difficulties may be encountered in practice, such as the complexity of market conditions, the availability of data, and the practical application effect of the model. Therefore, future research directions can further extend and improve the model, especially to verify its effectiveness in combination with actual market data. In addition, the application in different market environments and insurance products can be explored to further improve the universality and practicality of the model.

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